

Computations with transfer operators

Caroline Wormell

The University of Sydney

4.2 The spaces $W_p^{s,t}$ and $W_p^{s,\infty}$ 135

respect to some cone system; then we carry over the definition to the manifold M using suitable systems of charts and cones in \mathbb{R}^d from §4.2.1.

Let $\Theta = (\mathbf{C}_\xi, \Phi_\xi)$ be a cone system as in Definition 4.10. For real numbers l and s , recalling the function ψ_0 defined between (2.65) and (2.66), we introduce the symbols

$$\psi_{0,l}(\xi) = (1 + \|\xi\|^2)^{l/2} \psi_0\left(\frac{\xi}{\|\xi\|}\right) (1 - \psi_0(\xi)), \quad (4.17)$$

$$\psi_{0,s}(\xi) = (1 + \|\xi\|^2)^{s/2} \psi_0\left(\frac{\xi}{\|\xi\|}\right) (1 - \psi_0(\xi)). \quad (4.18)$$

(In the sequel we shall take $s < 0 < t$ with $t - s < r - 1$.) Just like in (2.67), we see that $\mathbb{F}^{-1}(a(\xi) \cdot \mathbb{F}(\varphi))(x)$, where \mathbb{F} is the Fourier transform (2.12)

Definition 4.16. Analogous to the Sobolev spaces $W_p^{s,t}$ and $W_p^{s,\infty}$ in \mathbb{R}^d , we define the Sobolev spaces $W_p^{s,t}$ and $W_p^{s,\infty}$ in \mathbb{R}^d , for compact $K \subset \mathbb{R}^d$, $1 \leq p < \infty$, and real numbers s, t .

$$\| \varphi \|_{W_p^{s,t}(K)} = \left(\int_{\mathbb{R}^d} |\psi_{0,s}(\xi)|^p |\mathbb{F}(\varphi)(\xi)|^p d\xi \right)^{1/p} \quad (4.19)$$

$$\| \varphi \|_{W_p^{s,\infty}(K)} = \sup_{\xi \in \mathbb{R}^d} |\psi_{0,s}(\xi)| |\mathbb{F}(\varphi)(\xi)| \quad (4.20)$$

(K be the support of φ .)

and let $\mathcal{L}_s \varphi(x) = (2+x)^{2s} \varphi\left(\frac{1}{2+x}\right)$

For $1 < p < \infty$, the analogue of Remark 2.12 gives

$$\| \varphi \|_{W_p^{s,t}(K)} \leq \| \mathcal{L}_s \varphi \|_{W_p^{s,t}(K)} \quad (4.21)$$

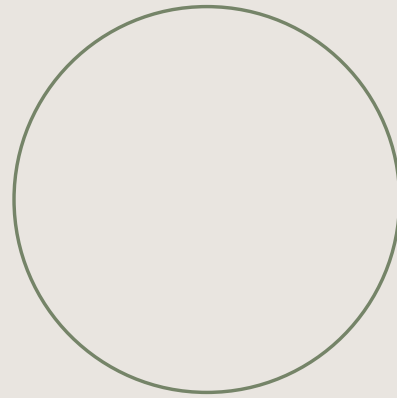
so that $(W_p^{s,t}(K))$ is isometric to $\{ \varphi \in W_p^{s,t}(\mathbb{R}^d) \mid \text{supp}(\varphi) \subset K \}$. The space $W_p^{s,t}(K)$ may be described in a similar (although not as neat) way via the injective (and surjective) map $(\psi_{0,s}^{(p)}, \psi_{0,t}^{(p)}, \psi_{0,\infty}^{(p)}) : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d) \oplus \mathcal{S}'(\mathbb{R}^d) \oplus \mathcal{S}'(\mathbb{R}^d)$. Since $(L_p)^* = L_{p/(p-1)}$ for $1 \leq p < \infty$, and the operators $\psi_{0,s}^{(p)}$ are self-adjoint, the dual of $W_p^{s,t}(K)$ is $W_{p/(p-1),s}^{s,t}(K)$. (We shall not need this.) The $s+s$ version of the norm is therefore more natural, in particular, for any $1 \leq p < \infty$ and $s \in \mathbb{R}$, we have

²⁵ In the original definition of [28, App. A], the multiplication by $(1 - \psi_0)$ had been universally omitted.

Circle geometry 😊

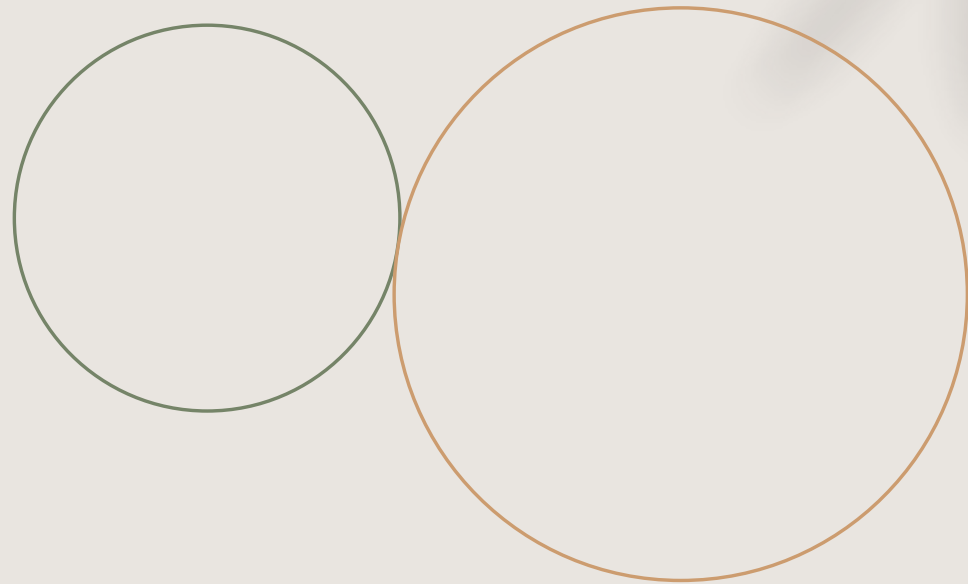
Circles

Theorem: it is possible to draw 1 circle



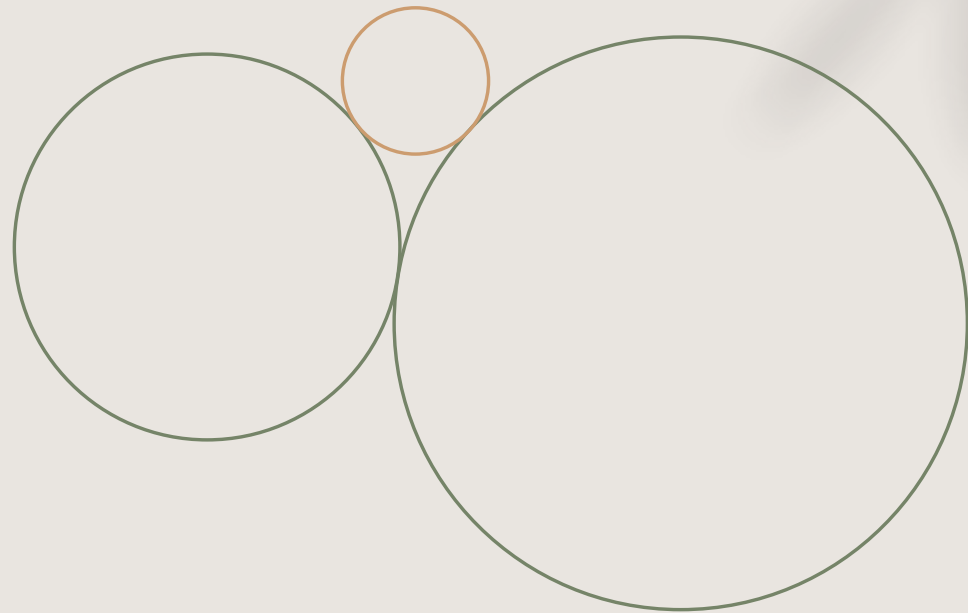
Circles

Theorem: Given any circle, it is possible to draw another one tangent to it



Circles

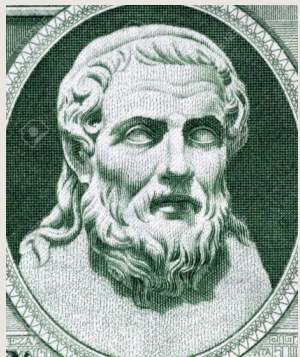
Theorem: Given any 2 tangent circles, it is possible to draw another one tangent to them both



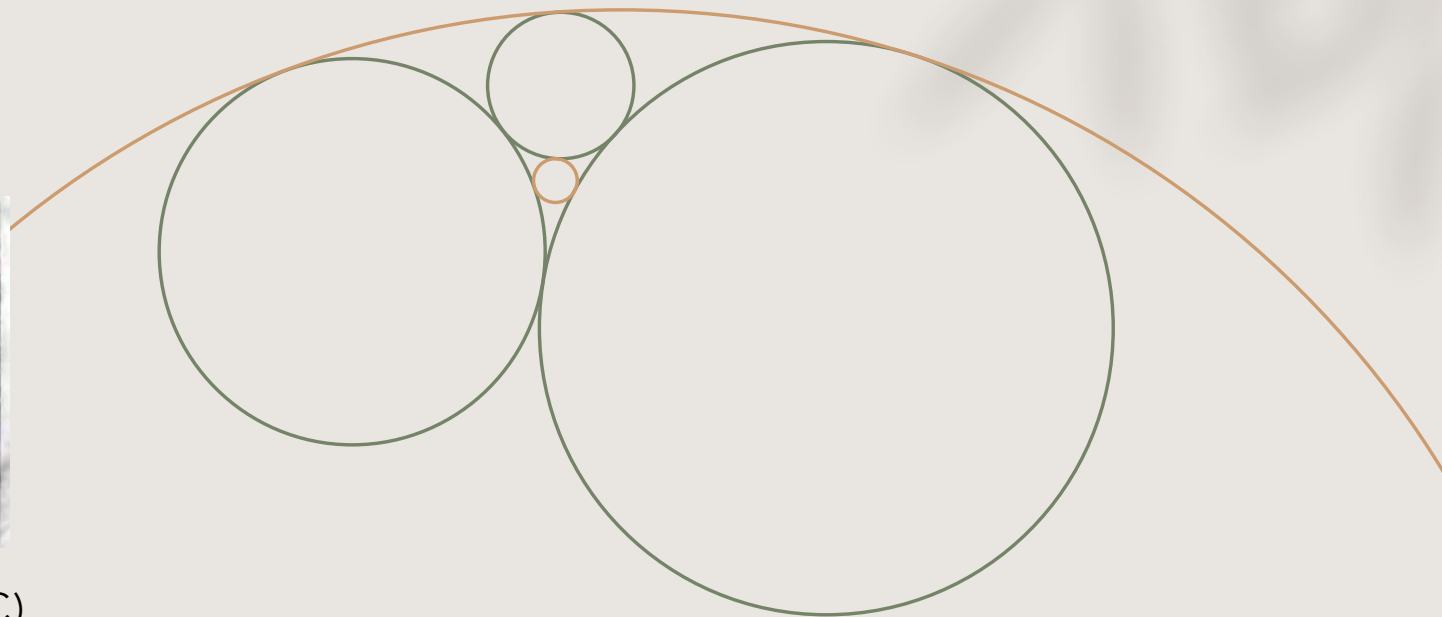
Circles

Theorem (Apollonius): Given any 3 tangent circles, it is possible to draw another one tangent to them all.

There are two choices for adding the fourth one.



Apollonius of Perga (c. 240-c. 190 BC)
was the inventor of conics



Circle packings

If you are bored, you can keep filling the gaps between the circles with other tangent circles:

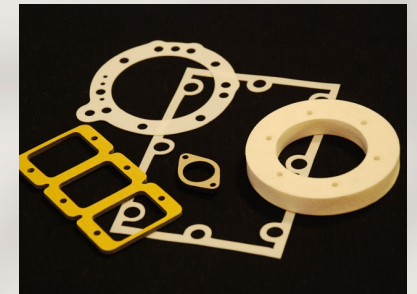
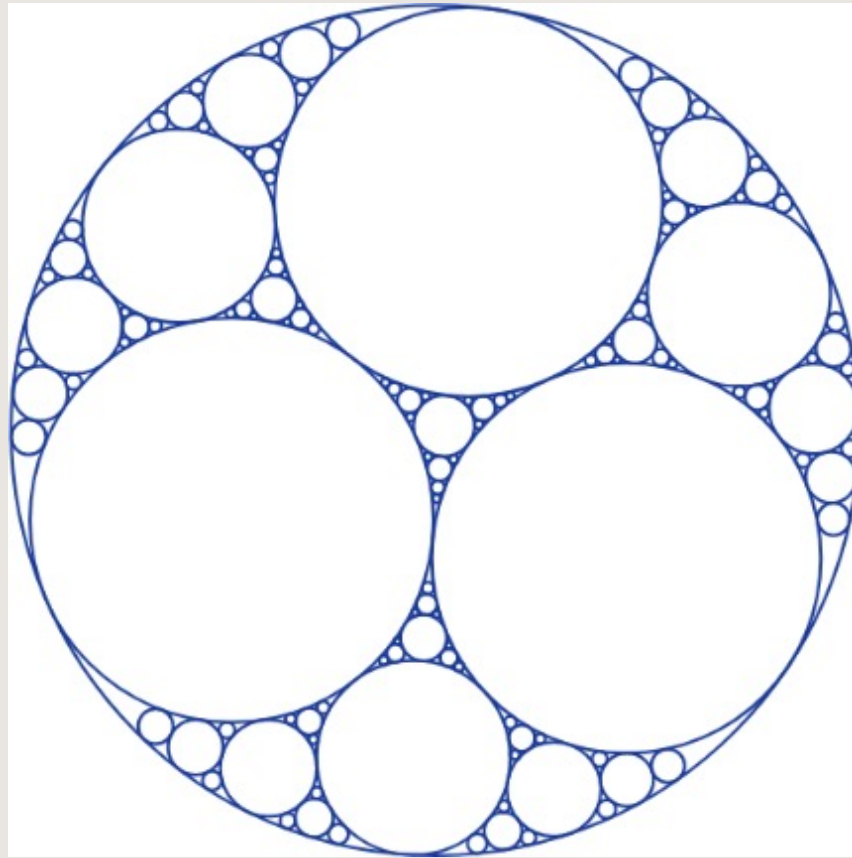


René Descartes
(1596-1650)



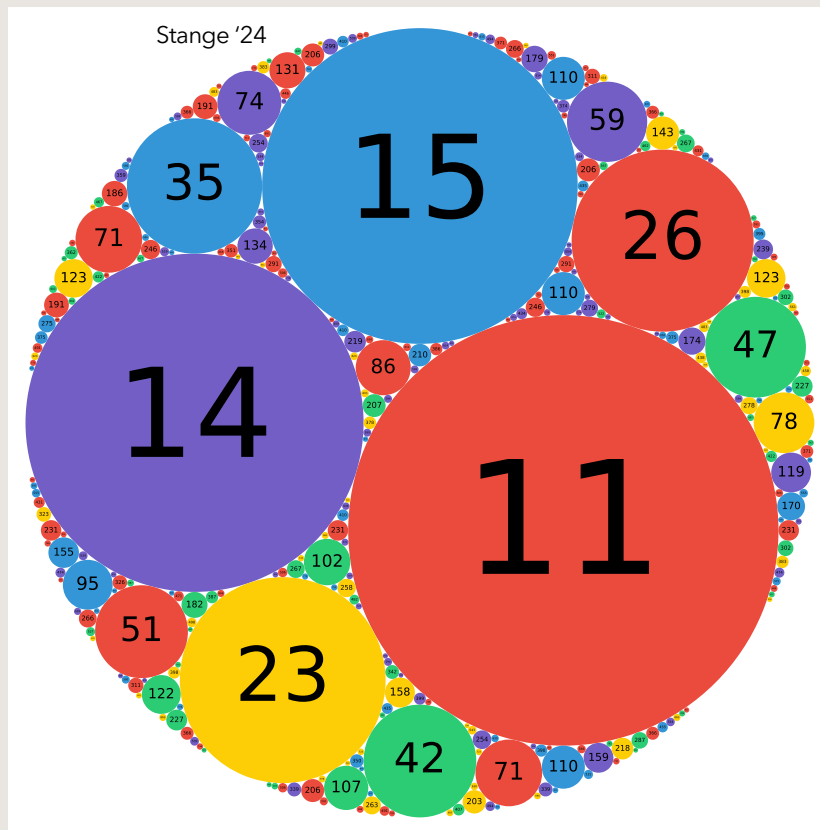
Seen in Smith's Alternative, Canberra

This gives you an Apollonian gasket



This gives you an Apollonian gasket

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2)$$



Outer circle = 6

Fact: if your first four circles have integer curvatures (=1/radius), so do the rest.

Counting circles leads to a lot of interesting number theory...

e.g. Curvatures must have certain sets of values mod 24 (Graham *et al.* '03)...

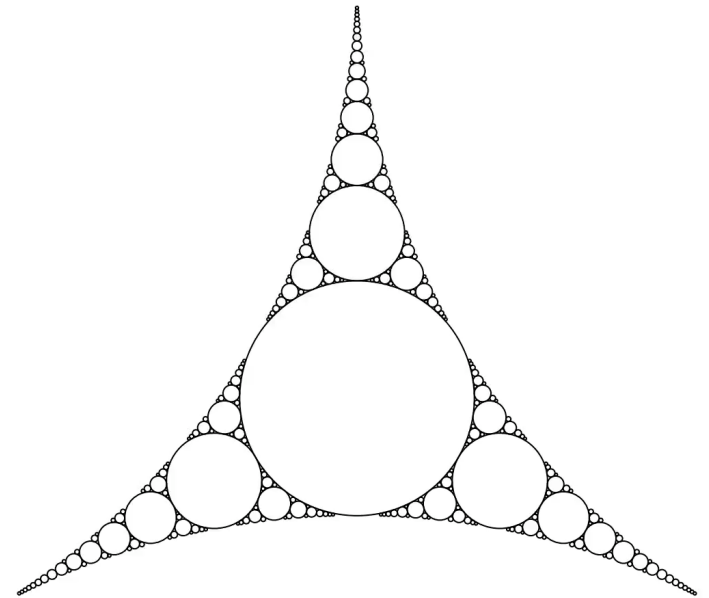
residues
0, 1, 4, 9, 12, 16
0, 5, 8, 12, 20, 21
0, 4, 12, 13, 16, 21
0, 8, 9, 12, 17, 20
3, 6, 7, 10, 15, 18, 19, 22
2, 3, 6, 11, 14, 15, 18, 23

with extra prohibitions on certain curvatures of the form an^2 , an^4 . (Haag *et al.* '24)

Key fact for computing:

Apollonian gaskets considered as subsets of \mathbb{C} are related by Möbius transformations

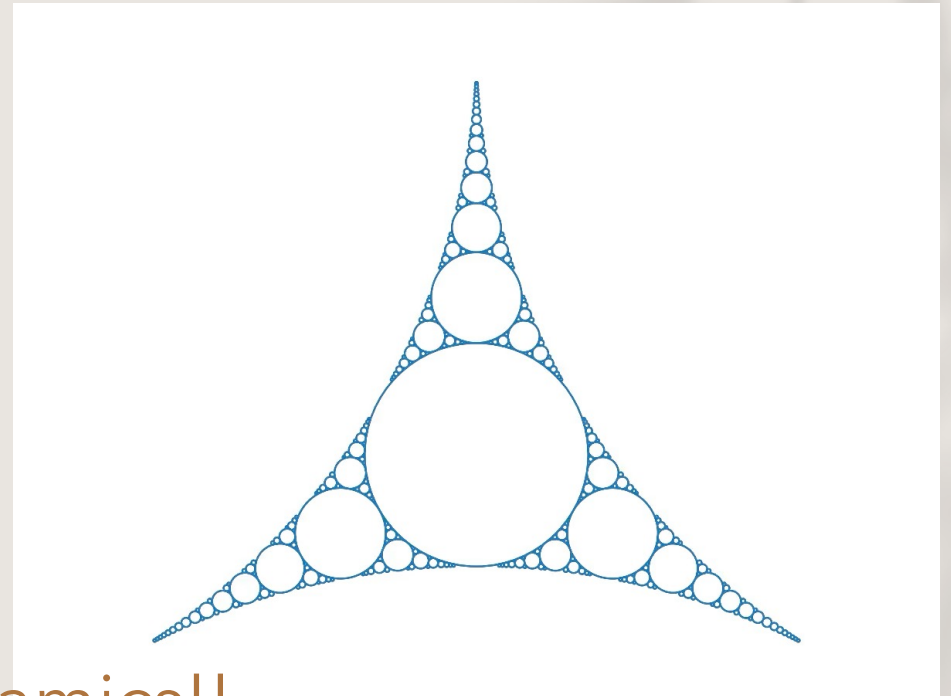
$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}$$



Key fact for computing

We can map the whole circular triangle into sub-circular triangles.

So the Apollonian gasket is an *invariant set* under some *maps*.



Dynamics!!

Q: How do we pin down “average” limiting behaviour associated to some nonlinear transformations?

A: Using **transfer operators**

Topic of this talk

A *transfer operator* is a weighted dynamical composition operator:

$$(\mathcal{L}\varphi)(x) = \sum_{i \in I} w_i(x) \varphi(T_i(x))$$

function $\varphi: X \rightarrow \mathbb{C}$

weights $w_i: X \rightarrow \mathbb{C}$

maps $T_i: X \rightarrow X$

This talk:

- Why transfer operators?
- How can you *compute* their properties with a high degree of certainty (rigour, accuracy)?
- What are some applications?

Why transfer operators?

The most obvious place to start in dynamics is in studying *invariant sets*.



These aren't linear objects, but signed measures on them are...

Transfer operator

Given transformations (say $T_i: X \rightarrow X$), we could define an operator that:

- divides up measures on X according to some functions $w_i > 0$ and
- pushes them forward by different transformations:

$$\mathcal{L}^* \mu := \sum_{i \in I} (T_i)_* (w_i \mu)$$

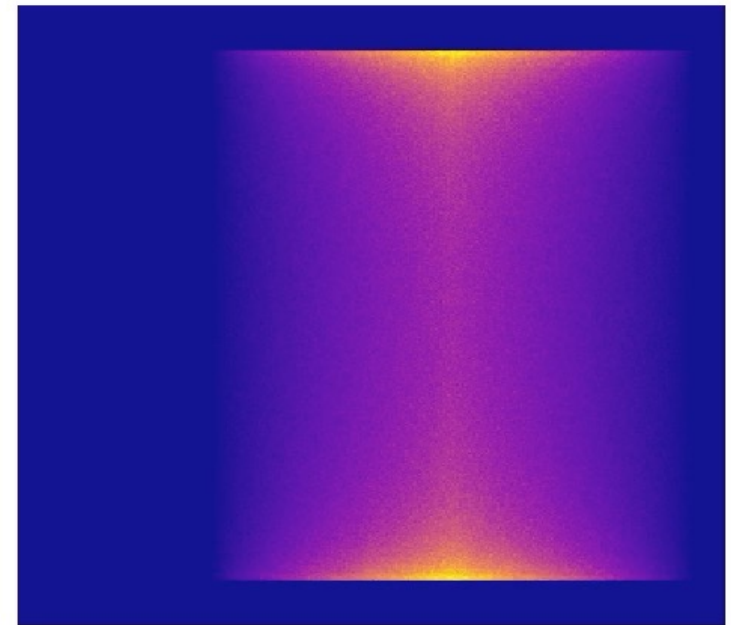
Consider an eigenmeasure ν with support Λ :

$$\mathcal{L}^* \nu = \lambda \nu$$

Now

$$\Lambda = \text{supp } \nu = \text{supp } \mathcal{L}^* \nu = \cup_i T_i(\Lambda)$$

so Λ is an invariant set!

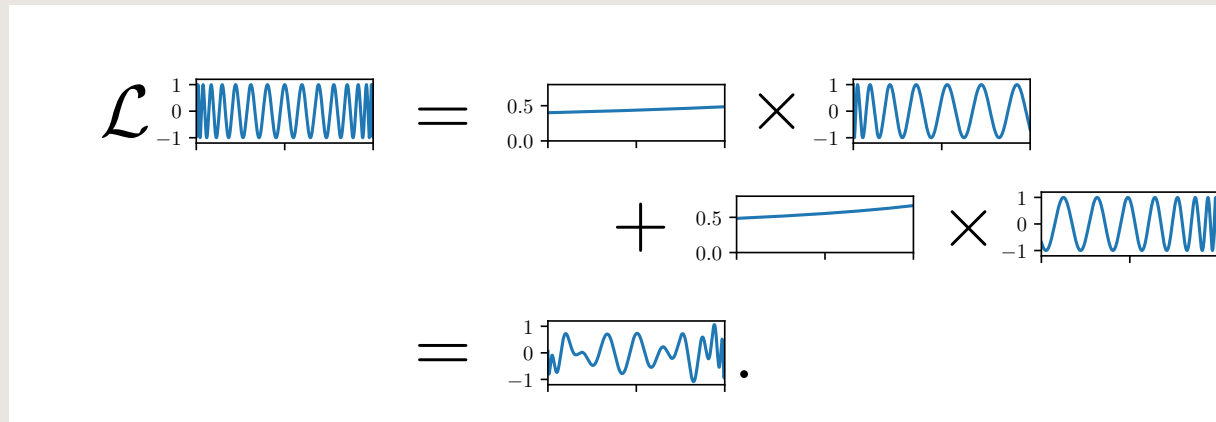


Duals

Studying the space of measures is a bit nasty, so we study the dual operator $\mathcal{L}: L^\infty(X) \rightarrow L^\infty(X)$

$$\mathcal{L}^* \mu := \sum_i T_{i*}(w_i \mu) \quad \leftrightarrow \quad \mathcal{L} \varphi = \sum_i w_i \varphi \circ T_i$$

This is much nicer as we are now dealing with functions!



Computation

- Most transfer operators don't produce analytic solutions (one exception: Blaschke products—see Cecilia's talk)
- To approximate them on a computer, we need:
 1. A Banach space where our transfer operator has some compactness properties (\Rightarrow stability under discretisation) **Difficult but a big industry**
 2. A sequence of finite-rank projections that converge quickly (i.e. a good discretisation) **In progress**
 3. A way to compute the action of the operator effectively. **Mostly(!) OK**
 4. Optionally, a way to *rigorously validate* your estimate. **Surprisingly good**

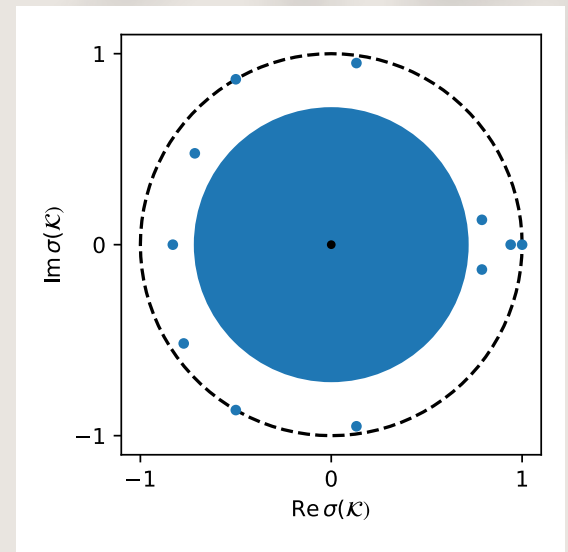
1. Stability to approximation

Many questions involving transfer operators (e.g. fractal dimension) concern an (often leading) eigenvalue.

Note:

- Point spectrum is much easier to approximate than continuous spectrum.
Work on continuous spectrum: Colbrook *et al.* '23 and descendents...
- However, we know that the discrete spectrum is (mostly) consistent across Banach spaces (Baladi and Tsuji '16)

Goal: find Banach space \mathcal{B} so that $\mathcal{L}: \mathcal{B} \rightarrow \mathcal{B}$ has discrete spectrum (at least somewhere).
“quasicompactness”



1. Stability to approximation

Goal: find a Banach space so that $\mathcal{L}: \mathcal{B} \rightarrow \mathcal{B}$ has discrete spectrum (at least where we are looking at it).

- For general maps T_i the Banach space \mathcal{B} needs to be tailored to the dynamics (e.g. functions with regularity along unstable manifolds, etc.)

132 4.2 Anisotropic Banach spaces defined via cones

4.2 The spaces $W_{p,r}^{s,\alpha}$ and $W_{p,r}^{s,\alpha}$

After introducing suitable finite systems of local charts and cones adapted to our hyperbolic map T in §4.2.1, we define the (microlocal) ‘Sobolev’ anisotropic spaces $W_{p,r}^{s,\alpha}(T, V)$ and $W_{p,r}^{s,\alpha}(T, V)$ in §4.2.2 (Definition 4.17). The ‘Hilbert’ anisotropic spaces $C_r^s(T, V)$ will be introduced in Definition 4.23 in Section 4.3.

4.2.1 Charts and cone systems adapted to (T, V)

In Section 4.2.2 (and Section 5.1.1), the hyperbolicity assumption will be used to define anisotropic spaces in charts, via a system of invariant cones for the cotangent dynamics. We introduce the relevant objects next. A cone in \mathbb{R}^d is a subset which is invariant under scalar multiplication. For two cones \mathbf{C} and \mathbf{C}' in \mathbb{R}^d , we write

$$\mathbf{C} \subseteq \mathbf{C}' \quad \text{if} \quad \overline{\mathbf{C}} \subseteq \text{interior}(\mathbf{C}') \cup \{0\}.$$

We say that a cone \mathbf{C} is d' -dimensional if $d' \geq 1$ is the maximal dimension of a linear subset of \mathbf{C} .

Definition 4.10 (Cone systems, $\Theta < \Theta'$). Let \mathbf{C}_1 and \mathbf{C}_2 be closed cones in \mathbb{R}^d with nonempty interiors, of respective dimensions d_1 and d_2 , and such that $\mathbf{C}_1 \cap \mathbf{C}_2 = \{0\}$ (i.e. the cones are transverse). Let $\Phi_s: \mathbb{S}^{d-1} \rightarrow [0, 1]$ be a C^∞ function on the unit sphere \mathbb{S}^{d-1} in \mathbb{R}^d satisfying

$$\Phi_s(\xi) = 1 \text{ if } \xi \in \mathbb{S}^{d-1} \cap \mathbf{C}_1, \quad \Phi_s(\xi) = 0 \text{ if } \xi \in \mathbb{S}^{d-1} \cap \mathbf{C}_2,$$

and define $\Phi_s: \mathbb{S}^{d-1} \rightarrow [0, 1]$ by

$$\Phi_s(\xi) = 1 - \Phi_s(\xi).$$

A quadruple $\Theta = (\mathbf{C}_1, \mathbf{C}_2, \Phi_s, \Phi_s)$ is called a cone system. For another such quadruple $\Theta' = (\mathbf{C}'_1, \mathbf{C}'_2, \Phi'_s, \Phi'_s)$, we write $\Theta < \Theta'$ if

$$(\mathbb{R}^d \setminus \mathbf{C}'_1) \subseteq \mathbf{C}_1, \quad (\mathbb{R}^d \setminus \mathbf{C}'_2) \subseteq \mathbf{C}_2, \quad (4.12)$$

(Condition (4.12) implies $\mathbf{C}_1 \in \mathbf{C}'_1$ and $\mathbf{C}_2 \in \mathbf{C}'_2$.)

Definition 4.11 (Cone-hyperbolic diffeomorphism). Let U be an open and bounded subset in \mathbb{R}^d , and let $\Theta = (\mathbf{C}_1, \mathbf{C}_2, \Phi_s, \Phi_s)$ and $\Theta' =$

4.2 The spaces $W_{p,r}^{s,\alpha}$ and $W_{p,r}^{s,\alpha}$

$(\mathbf{C}'_1, \mathbf{C}'_2, \Phi'_s, \Phi'_s)$ be two cone systems.¹³ A C^∞ diffeomorphism $F: U \rightarrow \mathbb{R}^d$ onto its image is cone-hyperbolic from Θ to Θ' if F^* extends to a bilipschitz C^0 diffeomorphism of \mathbb{R}^d such that

$$DF_x^*(\mathbb{R}^d \setminus \mathbf{C}_1) \subseteq \mathbf{C}'_1, \quad \forall x \in \mathbb{R}^d. \quad (4.13)$$

The sign \pm in the notation \mathbf{C}_i refers to the fact that we shall require (via the Paley–Littlewood Definition 4.17) Sobolev regularity with a positive exponent in the directions \mathbf{C}_1 , while considering Sobolev distributions with a negative exponent in the directions given by \mathbf{C}_2 . See Remark 4.13 and Definition 4.15.

Remark 4.12 (Choosing a larger/smaller cone system in the image/domain). If $\Theta' < \Theta$, then the identity map is cone-hyperbolic from Θ to Θ' . This remark will be used to obtain a Leiman lemma from the Lasota–Yorke Lemma 4.20 (see the proof of Lemma 4.20, and Corollary 5.10). However, in general we shall work with hyperbolic maps (recall (4.6–4.7)), and we may ensure that the image cone system is strictly larger. Indeed, if F is cone-hyperbolic from Θ to Θ' , then there exists a $\tilde{\Theta} < \Theta'$ such that F is cone-hyperbolic from $\tilde{\Theta}$ to Θ' , and there exists a $\tilde{\Theta}' > \Theta'$ such that F is cone-hyperbolic from Θ to $\tilde{\Theta}'$.

Remark 4.13 (Cone systems and stable/unstable cones). If F is hyperbolic, then by the previous remark there exist cone systems $\Theta < \Theta'$ such that F is cone-hyperbolic from Θ to Θ' . In that case, $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}'_1, \mathbf{C}'_2$ correspond to unstable cones for F , while $\mathbf{C}_1, \mathbf{C}_2$ correspond to stable cones for F . (See (4.10) below.) In the application to transfer operators $\mathcal{L}_p^s = \mathcal{L}_p^s \circ T^s$, the map F will be the extension to $\mathbb{R}^d \setminus \{0\}$ of a local branch of the hyperbolic diffeomorphism T . (In particular, composing with F improves regularity in the stable cone for F , which is reflected in the positive regularity exponent in the stable \mathbf{C}_1 in Definition 4.17.) In the literature [87, 88] one sometimes considers the transfer operator associated with $T^s \circ F^s$. In that case, the stable cone of T is an unstable cone for F and vice versa. In particular, \mathbf{C}_1 is an unstable cone for T .

For a C^∞ cone-hyperbolic diffeomorphism (onto its image) $F: U \rightarrow \mathbb{R}^d$ from $\Theta = (\mathbf{C}_1, \mathbf{C}_2, \Phi_s, \Phi_s)$ to $\Theta' = (\mathbf{C}'_1, \mathbf{C}'_2, \Phi'_s, \Phi'_s)$, and for a compact subset $K \subseteq U$, we put

$$\|F\|_{K, \Theta, \Theta'} = \sup_{x \in K} \sup_{\xi \in \mathbb{S}^{d-1}} \frac{|DF_x^*(\xi)|}{|\xi|}. \quad (4.14)$$

¹³ We denote the transpose matrix of A by A^t . We view $\mathbf{C}_{1,2}, \mathbf{C}'_{1,2}$ as locally constant cone fields on the compact bundle $\mathbb{P}U$ so that F^* acts on these cones via the tangent map DF .

¹⁴ Cone-hyperbolicity only depends on the data \mathbf{C}_1 and \mathbf{C}_2 .

4.2 The spaces $W_{p,r}^{s,\alpha}$ and $W_{p,r}^{s,\alpha}$

respect to some cone system; then we carry over the definition to the manifold M using suitable systems of charts and cones in \mathbb{R}^d from §4.2.1.

Let $\Theta = (\mathbf{C}_1, \Phi_s)$ be a cone system as in Definition 4.10. For real numbers α and s , recalling the function ψ_δ defined between (2.65) and (2.66), we introduce¹⁴ the symbols

$$\Phi_{s,\alpha}(\xi) = (1 + |\xi|^\alpha)^s \Phi_s\left(\frac{\xi}{|\xi|}\right) (1 - \psi_\delta(\xi)), \quad (4.17)$$

$$\Phi_{s,\alpha}(\xi) = (1 + |\xi|^\alpha)^s \Phi_s\left(\frac{\xi}{|\xi|}\right) (1 - \psi_\delta(\xi)). \quad (4.18)$$

(In the application, we shall take $s < 0 < t$ with $t - s < r - 1$). Just like in (2.73), we set $\mathcal{L}^s \psi_\delta(x) := \mathcal{F}^{-1}(\alpha(\xi)) \mathcal{F}(\psi_\delta(x))$, where \mathcal{F} is the Fourier transform (2.12).

Definition 4.16 (Anisotropic Sobolev spaces $W_{p,r}^{s,\alpha}$ and $W_{p,r}^{s,\alpha}$). In \mathbb{R}^d , for a compact set $K \subseteq \mathbb{R}^d$ with nonempty interior, $1 \leq p < \infty$, and real numbers s, t , set for $\varphi \in C^\infty(K)$

$$\|\varphi\|_{W_{p,r}^{s,\alpha}(K)} = \|\Phi_{s,\alpha}^{p/r}(\varphi) + \psi_\delta^{p/r}(\varphi)\|_{L^p}, \quad (4.19)$$

$$\|\varphi\|_{W_{p,r}^{s,\alpha}(K)} = \|\Phi_{s,\alpha}^{p/r}(\varphi)\|_{L^p} + \|\psi_\delta^{p/r}(\varphi)\|_{L^p}. \quad (4.20)$$

Then let $W_{p,r}^{s,\alpha}(K)$ be the completion of $C^\infty(K)$ with respect to $\|\cdot\|_{W_{p,r}^{s,\alpha}(K)}$, and let $W_{p,r}^{s,\alpha}(K)$ be the completion of $C^\infty(K)$ with respect to $\|\cdot\|_{W_{p,r}^{s,\alpha}(K)}$.

The operator $\Phi_{s,\alpha}^{p/r} + \psi_\delta^{p/r}: \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ is bijective on $\mathcal{S}'(\mathbb{R}^d)$ and we may define

$$W_{p,r}^{s,\alpha}(\mathbb{R}^d) := (\Phi_{s,\alpha}^{p/r} + \psi_\delta^{p/r})^{-1}(L_p(\mathbb{R}^d)),$$

with induced norm. For $1 < p < \infty$, the analogue of Remark 2.12 gives

$$W_{p,r}^{s,\alpha}(K) = \{ \varphi \in \mathcal{S}'(\mathbb{R}^d) \mid \text{supp}(\varphi) \subseteq K, \|\Phi_{s,\alpha}^{p/r}(\varphi) + \psi_\delta^{p/r}(\varphi)\|_{L^p(K)} < \infty \}, \quad (4.21)$$

so that $W_{p,r}^{s,\alpha}(K)$ is isometric to $\{ \varphi \in W_{p,r}^{s,\alpha}(\mathbb{R}^d) \mid \text{supp}(\varphi) \subseteq K \}$. The space $W_{p,r}^{s,\alpha}(K)$ may be described in a similar (although not as neat) way via the bijective (non surjective) map $\Phi_{s,\alpha}^{p/r} + \psi_\delta^{p/r}: \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d) \otimes \mathcal{S}'(\mathbb{R}^d)$. Since $(L_p)^* = L_{p/(p-1)}$ for $1 \leq p < \infty$, and the operators $\Phi_{s,\alpha}^{p/r}$ are self-adjoint, the dual of $W_{p,r}^{s,\alpha}(K)$ is $W_{p/(p-1),r}^{s,\alpha}(K)$. (We shall not need this.) The r -norm version of the norm is therefore more natural, in particular, for any $1 \leq p \leq \infty$ and all $t \in \mathbb{R}$, we have

¹⁵ In the original definition of [88, App. A], the multiplication by $(1 - \psi_\delta)$ had been tacitly omitted.

136 4.2 Anisotropic Banach spaces defined via cones

$$W_{p,r}^{s,\alpha}(K) = L_p(K), \quad W_{p,r}^{s,\alpha}(K) = H_p^s(K). \quad (4.22)$$

(See also Proposition 4.20, as well as the Comments at the end of this chapter.) This double-norm norm is also useful to manipulate to prove Lasota–Yorke inequalities when T is C^∞ (see Appendix D.4). However, the norm $W_{p,r}^{s,\alpha}$ is easier to describe in Paley–Littlewood form (see (4.20) and Proposition 4.21). We shall therefore use $W_{p,r}^{s,\alpha}$ to prove the Lasota–Yorke estimate in §4.3.1.

Definition 4.17 (Anisotropic spaces $W_{p,r}^{s,\alpha}$ and $W_{p,r}^{s,\alpha}$ on M). Fix $1 < p < \infty$, and real numbers s and t . Fix C^∞ charts $\kappa_\nu: U_\nu \rightarrow \mathbb{R}^d$ a partition of unity θ_ν and cone systems Θ_ν satisfying the requirements of Definitions 4.14 and 4.15 in Section 4.2.1. The Banach spaces $W_{p,r}^{s,\alpha}(T, V)$ and $W_{p,r}^{s,\alpha}(T, V)$ are the completion of $C_c^\infty(\mathbb{T})$ for the respective norms¹⁶

$$\|\varphi\|_{W_{p,r}^{s,\alpha}(T, V)} := \max_{\nu \in \mathcal{I}} \|(\theta_\nu \varphi) \circ \kappa_\nu^{-1}\|_{W_{p,r}^{s,\alpha}(\mathbb{R}^d)},$$

and

$$\|\varphi\|_{W_{p,r}^{s,\alpha}(T, V)} := \max_{\nu \in \mathcal{I}} \|(\theta_\nu \varphi) \circ \kappa_\nu^{-1}\|_{W_{p,r}^{s,\alpha}(\mathbb{R}^d)},$$

Remark 4.18. It is not explicit in our notation, but the spaces $W_{p,r}^{s,\alpha}(T, V)$ depend on the system of charts $\{(\kappa_\nu, \theta_\nu)\}$, the cone systems $\{(\mathbf{C}_{\nu,1}, \nu_{\nu,1})\}$, and the partition of unity $\{\theta_\nu\}$. Choosing a different system of local charts, a different set of cone systems, or a different partition of unity, does not a priori give rise to equivalent norms, although the Lasota–Yorke bounds in Lemma 4.20 give relations. This does not cause problems.

We shall see in the proof of Theorem 4.6 that the next proposition implies that, for the purposes of studying the essential spectral radius of the transfer operator, the s and r norms can be viewed as equivalent.

Proposition 4.19 (Comparing $W_{p,r}^{s,\alpha}$ and $W_{p,r}^{s,\alpha}$). For each $1 < p < \infty$, any $\Theta < \Theta'$ and every $t, s \in \mathbb{R}$ there exists a C such that

$$\|\varphi\|_{W_{p,r}^{s,\alpha}} \leq \|\varphi\|_{W_{p,r}^{s,\alpha}} \leq C \|\varphi\|_{W_{p,r}^{s,\alpha}}, \quad \text{for } \varphi \in C^\infty(K). \quad (4.23)$$

Proof. The first inequality is trivial. To prove the second inequality in (4.23), it is enough to show

$$\max\{\|\Phi_{s,\alpha}^{p/r}(\varphi)\|_{L^p}, \|\psi_\delta^{p/r}(\varphi)\|_{L^p}\} \leq C \|\Phi_{s,\alpha}^{p/r}(\varphi) + \psi_\delta^{p/r}(\varphi)\|_{L^p}$$

for some constant C . Since

¹⁶ By Lemma 4.14, we can equivalently take the completion of $C_c^\infty(\mathbb{T})$. In addition, the same comment as in the footnote to Definition 2.11 applies here.

Some Banach spaces for uniformly hyperbolic dynamics (Baladi '18)

1. Stability to approximation

Note Apollonian circle packing is 2 real dimensions

Super ezy mode: uniform **1D** contractions:

$$T_i: [-1,1] \rightarrow [-1,1] \text{ s.t. } \sup_{x \in [-1,1]} |T_i'(x)| \leq \gamma < 1.$$

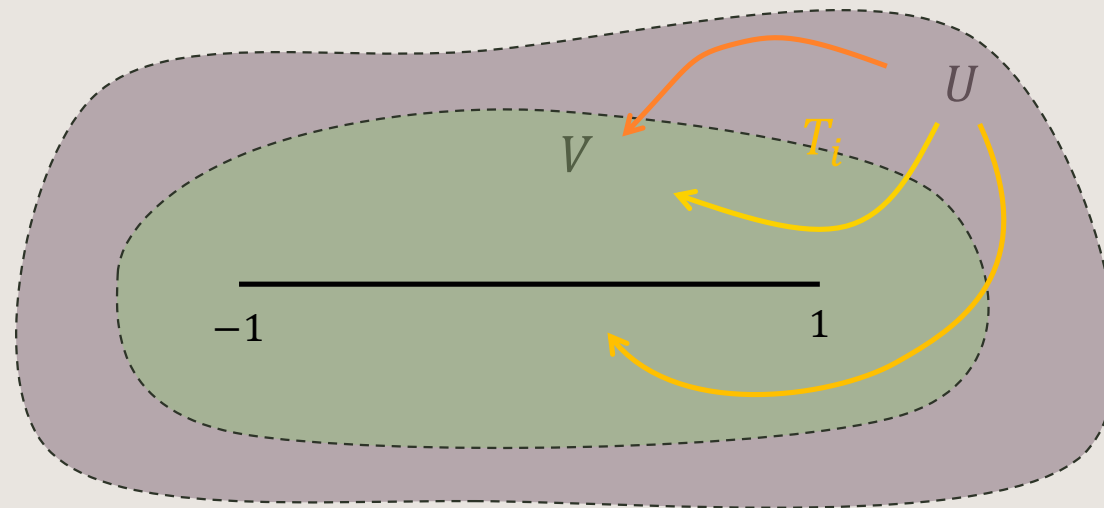
- Almost any space of at least Hölder functions will give us a quasi-compact operator.
- But we want the best for computation (=“most compact”)

1. Stability to approximation: Banach space

Let's imagine that there are some open sets U, V with

$$[-1,1] \subset U \subseteq \bar{U} \subset V \subseteq \mathbb{C}$$

so that the T_i, w_i extend analytically to U so each T_i maps U into V .



1. Stability to approximation: Banach space

Define the Hardy space:

$$H(U) = \{\varphi: U \rightarrow \mathbb{C} \text{ analytic and bounded}\}$$

with the sup-norm on U .

$$\varphi(z) = \sum_i w_i(z) \varphi(T_i(z))$$

Then:

- $\mathcal{L}: H(V) \rightarrow H(U)$ is bounded
- $H(V)$ embeds compactly in $H(U)$.

So $\mathcal{L}: H(U) \rightarrow H(U)$ is compact and thus approximable in operator norm by finite rank operators

2. Discretisation: polynomials

Let's try and project our operator \mathcal{L} onto a space of polynomials:

$$V = \{\text{polynomials of degree} \leq K\}$$

There are many ways to do this of varying quality.

Given K points $\{x_j\}$ we can choose as a basis of V the *Lagrange polynomials*, defined by:

$$\ell_j(x_{j'}) = \begin{cases} 1, & j = j' \\ 0, & \text{else} \end{cases}$$

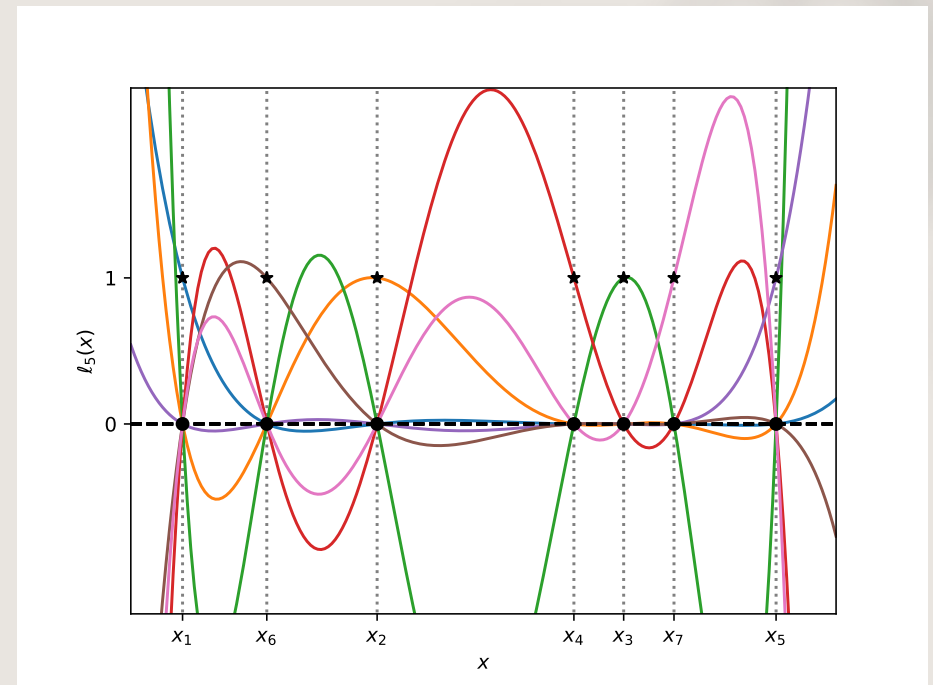
$$\text{(so } \ell_j(x) = \prod_{j' \neq j} \frac{x - x_{j'}}{x_j - x_{j'}} \text{)}$$

2. Discretisation: polynomials

We can use Lagrange polynomials to interpolate a function:

$$\varphi(x) \approx \sum_{j=1}^K \varphi(x_j) \ell_j(x)$$

For most choices of points $\{x_j\}$, this is a terrible idea.



2. Discretisation: polynomials

But for Chebyshev points, it works great:

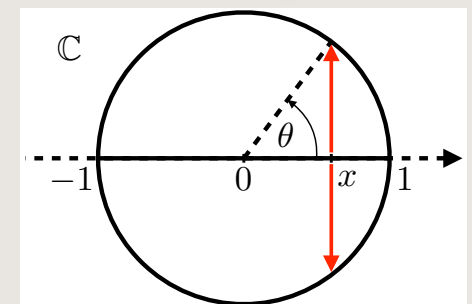
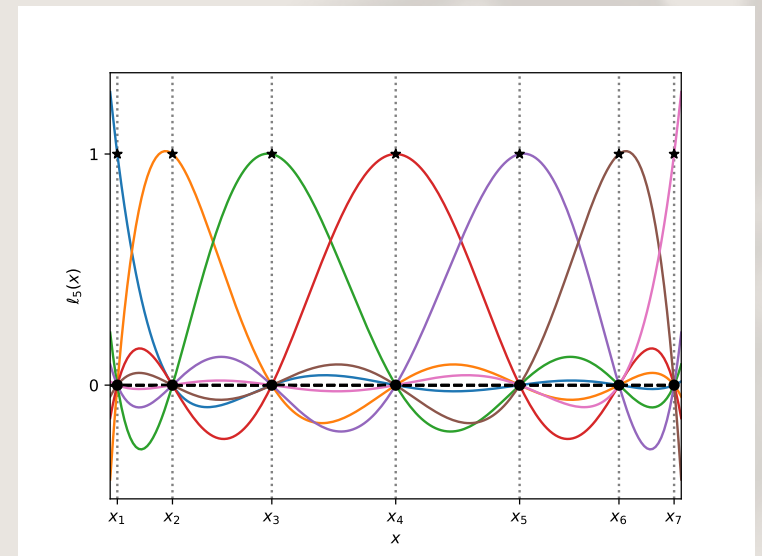
$$x_k = \cos \frac{\pi(2k-1)}{2k} \in [-1,1]$$

Why?

$$x \rightarrow \cos \theta$$

Chebyshev points \rightarrow evenly spaced points

polynomials \rightarrow trig functions



2. Discretisation: polynomials

We can interpolate \mathcal{L} in the polynomial basis:

$$\mathcal{L}^{(K)} \ell_k = \sum_{j=1}^K \ell_j (\mathcal{L} \ell_k)(x_j)$$

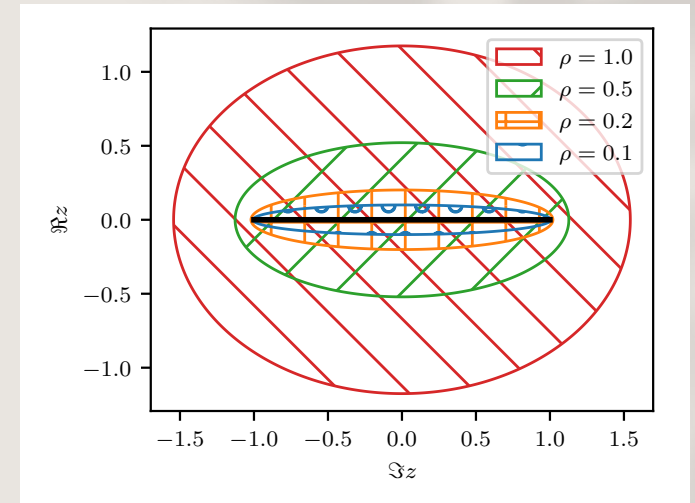
Now, suppose our sets U, V are *Bernstein ellipses*:

Theorem (Bandtlow & Slipantschuk '19, 1D; W. and Vytova '25 nD)

Let $\mathcal{L}^{(K)}$ be the K -point interpolation of \mathcal{L} . Then there exist (constructible) constants C, c

$$\|\mathcal{L}^{(K)} - \mathcal{L}\|_{H(U)} \leq C e^{-cK}$$

All our estimates are exponentially good!



$$\cos(\mathbb{R} + i[-\rho, \rho])$$

Work using Chebyshev discretisations

- Lyapunov exponents (**W. '19**, W. '21, Pollicott-Vytnova '23), statistical laws (W. '19, Crimmins-Froyland '19), metric entropy (Pollicott-Slipantschuk '24)
- Eigenvalues, almost-invariant sets (**Bandtlow-Slipantschuk '19**, Blumenthal *et al.* '25)
- Hausdorff dimension (Pollicott-Vytnova '22 , **Vytnova-W. '25**)
- Selberg zeta functions (Bandtlow *et al.* '21)
- Lagrange and Markov spectra (Pollicott-Vytnova '22, Matheus-Moreira-Vytnova '22)
- Fourier transform of fractal measures (W. '23)
- Linear response problems (Nisoli-Taylor-Crush '23, Froyland-Galatolo '23)
- Extended Dynamical Mode Decomposition (Bandtlow-Just-Slipantschuk group '23-, W. 25, Herwig *et al.* '25, ...)

Application 1: fractal Hausdorff dimension

If there exists a *Frostman measure* μ supported on a set Λ with

$$\mu(B(x, r)) \leq Cr^s \text{ for all } x \in \Lambda$$

then the Hausdorff dimension of Λ is greater than or equal to s .

Under some nice conditions (T_i conformal...) we can find such measures as eigenmeasures of a transfer operator ($\mathcal{L}_s^* \mu = \mu$)

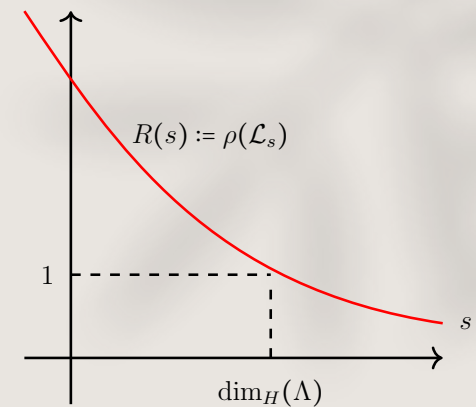
$$(\mathcal{L}_s \psi)(x) := \sum_{i \in I} |DT_i(x)|^s \psi(T_i(x))$$

Application 1: fractal Hausdorff dimension

Suppose the fractal Λ is generated by uniform conformal, non-overlapping contractions $\{T_i\}$. Define

$$(\mathcal{L}_s \psi)(x) := \sum_{i \in I} \underbrace{|DT_i(x)|^s}_{<1} \psi(T_i(x))$$

- $R(s) := \text{specrad } \mathcal{L}_s$ is strictly decreasing
- $R(\dim_H(\Lambda)) = 1$ (Ruelle-Bowen formula)



Application 1: fractal Hausdorff dimension

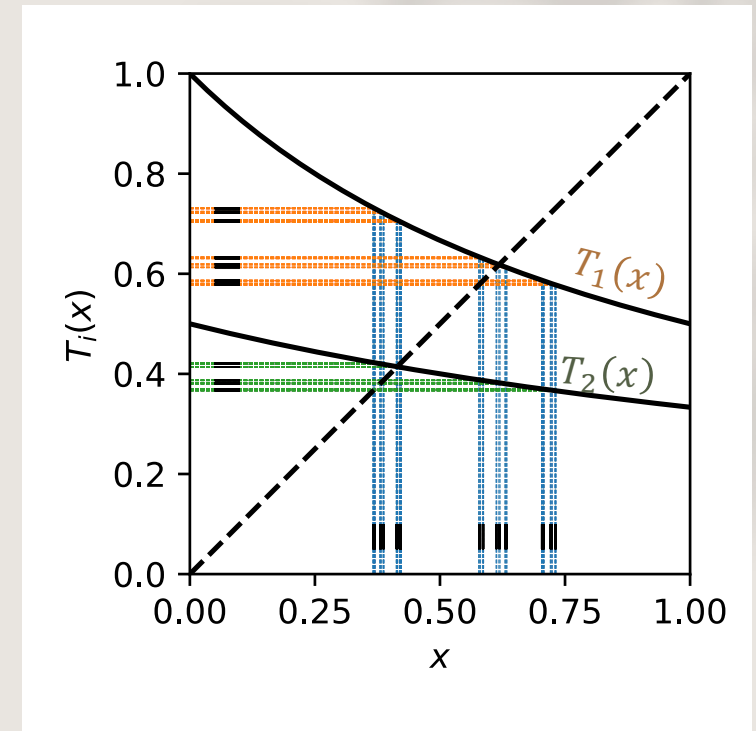
Numbers can have interesting *continued fractions*.

For example, they can just contain 1 and 2, e.g.

$$x = \frac{1}{2 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \dots}}}}}$$

These numbers form a set $E_2 \subset [0,1]$ invariant under the contractions*

$$T_1(x) = \frac{1}{1+x}, T_2(x) = \frac{1}{2+x}$$



Application 1: fractal Hausdorff dimension

The issue is basically to find $s \in [0,1]$ such that 1 is the leading eigenvalue of

$$\mathcal{L}_s \varphi(x) = \frac{1}{(1+x)^{2s}} \varphi\left(\frac{1}{1+x}\right) + \frac{1}{(2+x)^{2s}} \varphi\left(\frac{1}{2+x}\right)$$

Theorem 1.5.³

$\dim_H(E_2)$

$$\begin{aligned} &= 0.5312805062\ 7720514162\ 4468647368\ 4717854930\ 5910901839\ 8779888397 \\ &\quad 8039275295\ 3564383134\ 5918109570\ 1811852398\ 8042805724\ 3075187633 \\ &\quad 4223893394\ 8082230901\ 7869596532\ 8712235464\ 2997948966\ 3784033728 \\ &\quad 7630454110\ 1508045191\ 3969768071\ 3 \pm 10^{-201}. \end{aligned}$$

Details on the proof of this bound appear in §4.1.2. Whereas it may not be clear why a knowledge of $\dim_H(E_2)$ to 200 decimal places is beneficial, it at least serves to illustrate the effectiveness of the method we are using compared with earlier approaches.

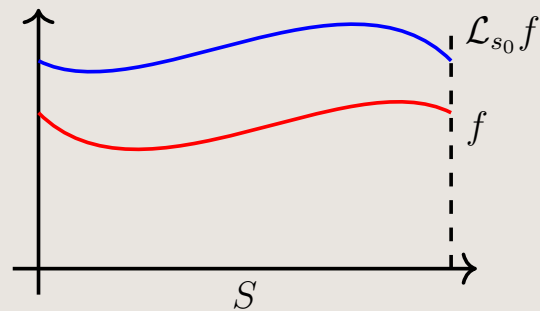
Pollicott and Vytnova, TAMS, 2022

More advanced applications to Markov and Lagrange spectra, Zaremba conjecture...

Rigorous validation

To turn your computer bound into a theorem, you need:

- Interval arithmetic to keep track of round-off errors and 1D approximations (e.g. truncating sums)
- A trick to go from finite dimensions to the full problem
I used to loathe this part, but if you have the *right trick* it is not that bad
The innovation of Pollicott-Vytnova '22 was to harness the fact that \mathcal{L}_s are positive operators



Application 1a: Apollonian circle packing

- For half a century the bound for the Apollonian gasket's dimension was:

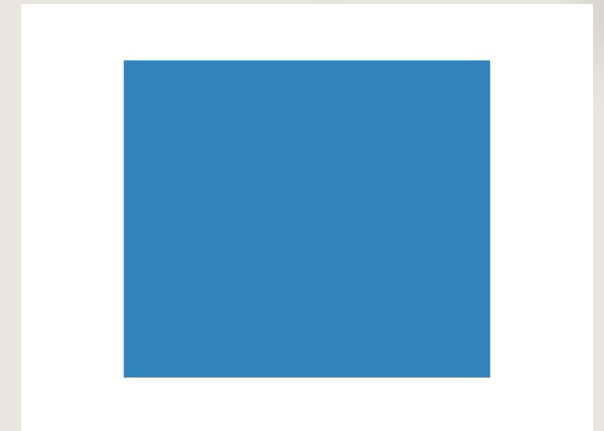
Theorem (Boyd, 1973):
 $1.300197 < \alpha < 1.314534$

- Various non-rigorous estimates using Ruelle-Bowen formula:
Thomas and Dhar '94, Curtis McMullen '98, de Leo '14, Bai-Finch '18

Issues:

- The contractions are actually non-uniform
- They didn't have much control over their discretisations

Joint work with Polina Vytnova (University of Surrey)



Application 1a: Apollonian circle packing

- For half a century the bound for the Apollonian gasket's dimension was:

Theorem (Boyd, 1973):
 $1.300197 < \alpha < 1.314534$

- Various non-rigorous estimates using Ruelle-Bowen formula:
Thomas and Dhar '94, Curtis McMullen '98, de Leo '14, Bai-Finch '18

Issues:

- The contractions are actually non-uniform
- They didn't have much control over their discretisations

Joint work with Polina Vytnova (University of Surrey)



Non-uniform contractions

Our strategy* is to combine maps to get uniform contractions.

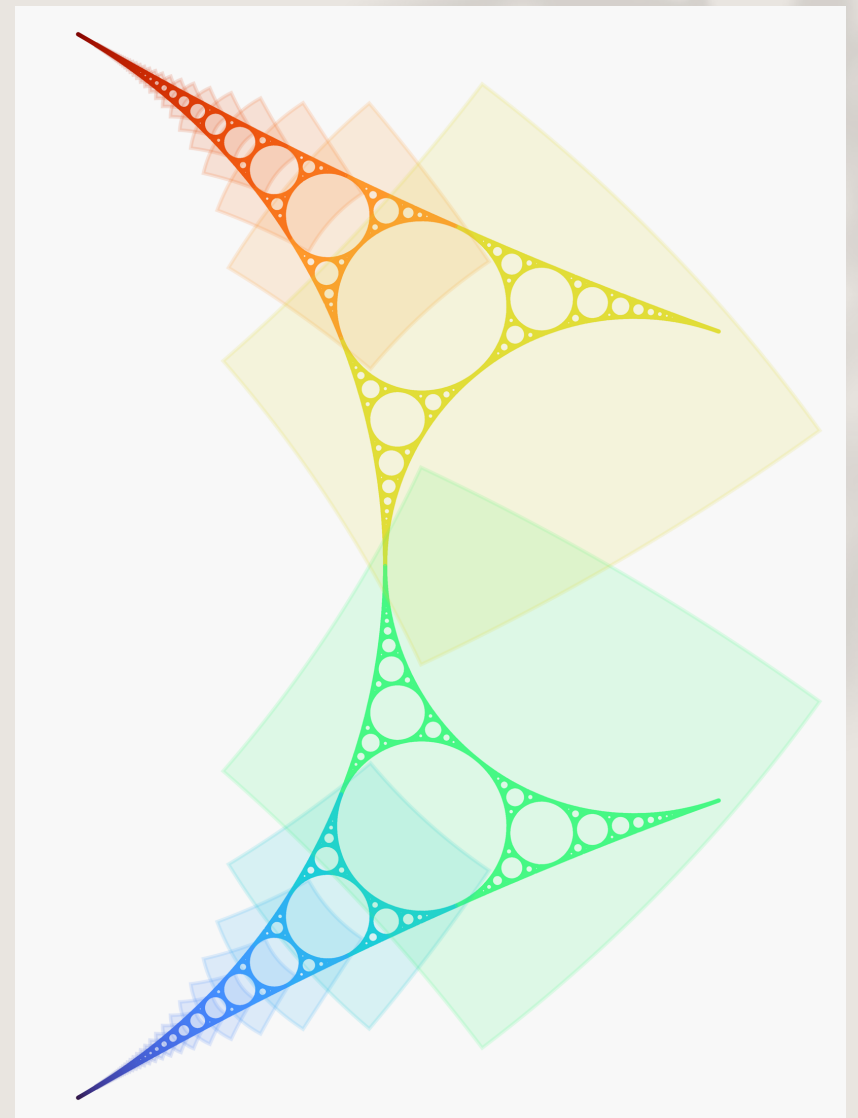
This led to an infinite number of branches:

$$(\mathcal{L}_s \varphi)(x) = \sum_{\substack{n=1 \\ \pm \in \{+, -\}}}^{\infty} \underbrace{J_{n,\pm}(x)^s \varphi(T_{n,\pm}(x))}_{O(n^{-2s})}$$

But we can approximate the tail via Euler-Maclaurin formula...

$$\sum_{n=N}^{\infty} J_{n,\pm}(x)^s \varphi(T_{n,\pm}(x)) \sim \sum_{l=0}^{\infty} c(l, s) N^{1-2s-l}$$

* probably the most computationally effective approach for any low-dimensional dynamics **!?**



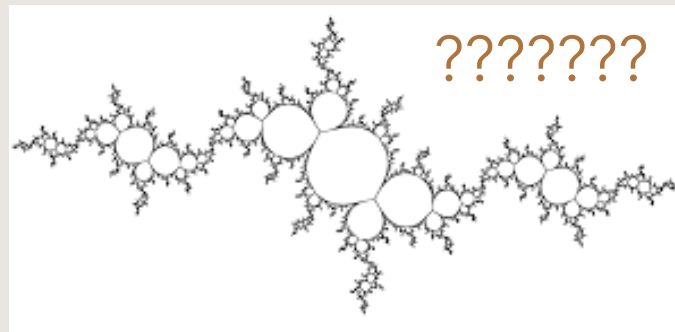
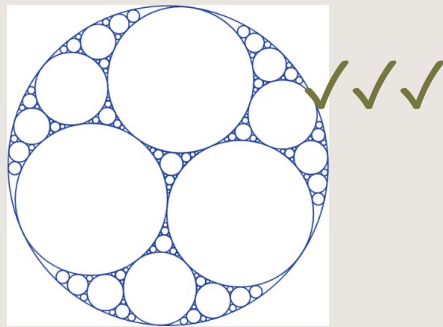
Application 1a: Apollonian circle packing

Problem of computation “solved”:

Theorem (Vytnova and W., *Invent. Math.* 2025)

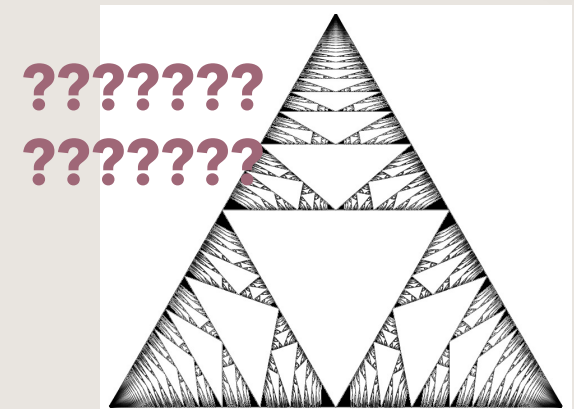
$$\alpha = 1.3056867280\ 4987718464\ 5986206851\ 0408911060\ 2644149646$$
$$8296446188\ 3889969864\ 2050296986\ 4545216123\ 1505387132$$
$$8079246688\ 2421869101\ 967305643 \pm 10^{-129}$$

- In this algorithm, resources $\sim (\#\text{certified digits})^{5+\epsilon}$.
- The same ideas transfer to a lot of other fractals!



Parabolic Julia sets

Rauzy gasket (from interval exchange transformations)



Application 2: Koopman operators

Suppose you have a dynamical system $T: M \rightarrow M$ and you are interested in the evolution of states (Fokker-Planck operator but deterministic).

That involves studying

$$\mathcal{K}^* \mu := T_* \mu$$

or its adjoint, known as the Koopman operator:

$$\mathcal{K} \psi := \psi \circ T$$

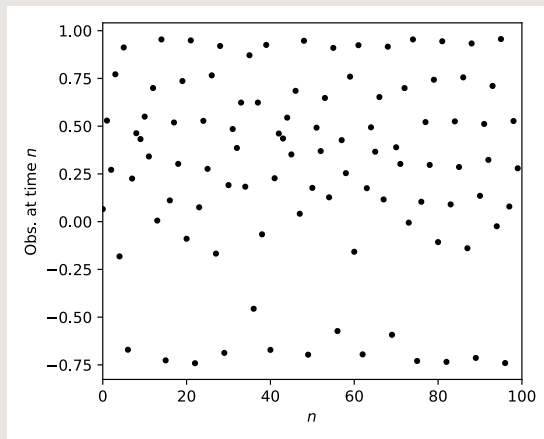
The eigenfunctions of this operator identify invariant sets, almost-invariant sets...

Application 2: Koopman operators

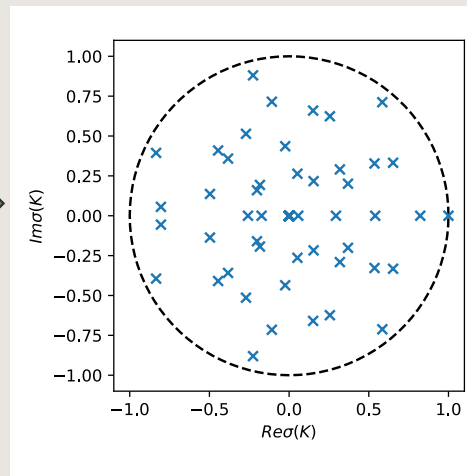
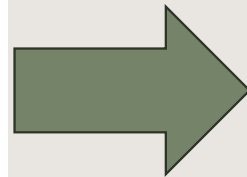
$$\mathcal{K}\psi := \psi \circ T$$

Hope: study Koopman operators from observations:

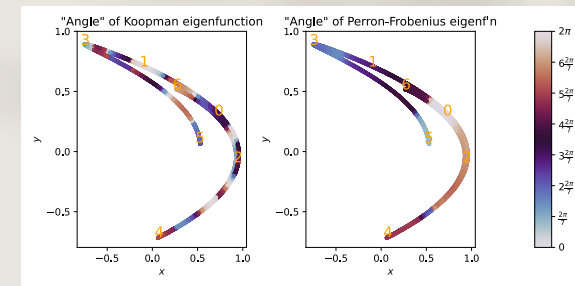
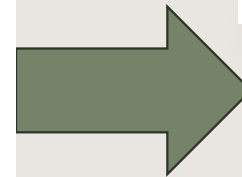
$$(\mathcal{K}\psi)(x_k) = T(x_k) = x_{k+1}$$



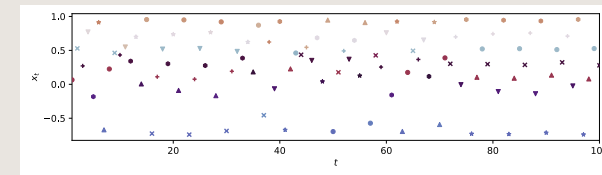
Time series (maybe high-dimensional/partially observed/...)

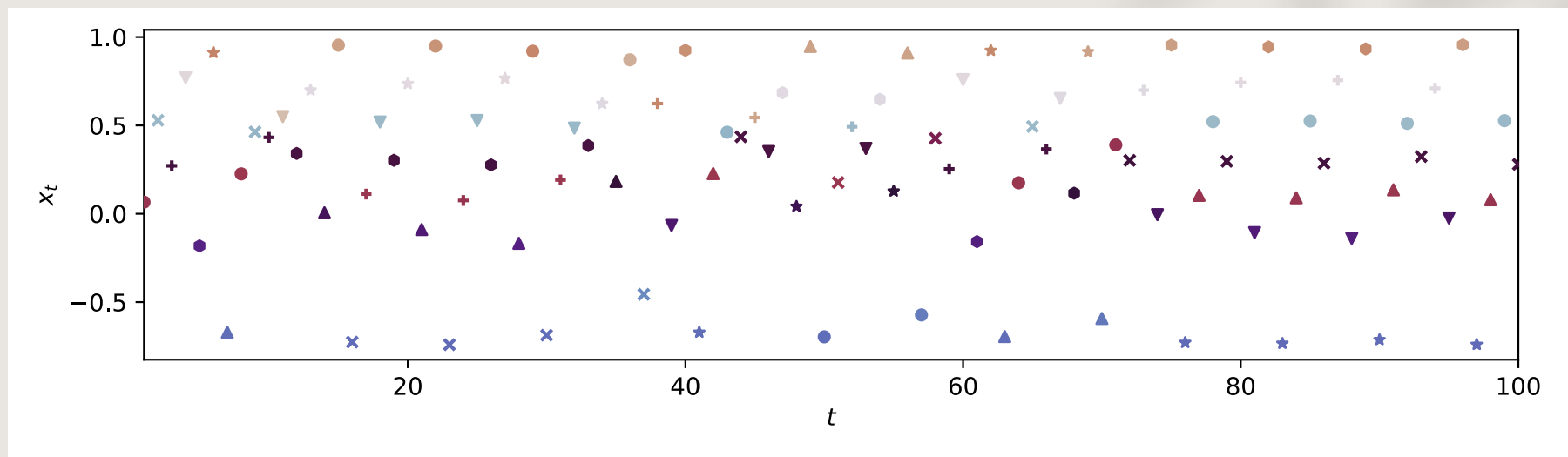
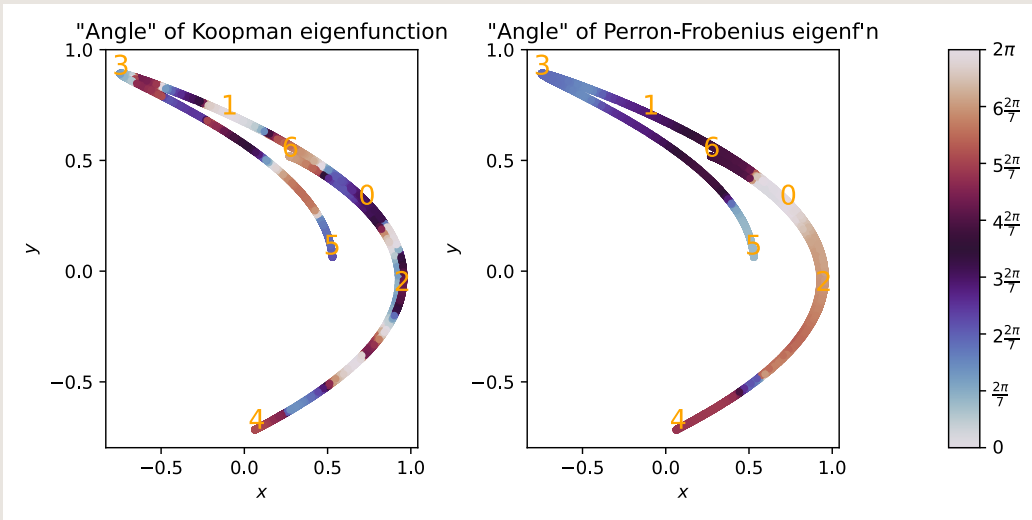


$\sigma(K)$



Almost invariant sets, metastable structures...





Application 2: Koopman operators

All our ideas are the same but with different basis functions.

Given function basis $\{\psi_0, \psi_1, \dots, \psi_B\}$ find matrix K that minimizes least squares error:

$$K\mathbf{v} = \operatorname{argmin}_{\substack{\text{p deg. } \leq n \\ \text{trig.poly.}}} \sum_{m=1}^M \left| \sum_b (K\mathbf{v})_b \psi_b(x_m) - \sum_b \mathbf{v}_b \underbrace{\psi_b(T(x_m))}_{\mathcal{K}\psi_b(x_m)} \right|^2$$

Application 2: Koopman operators

Huge industry, and it depends on your basis functions:

- piecewise constant functions (= Ulam's method) (Dellnitz *et al.* 2001 - 300+ citations)
- Via linear functions/delay variables (DMD) (Tu '13, 2600+ citations)
- Via low-order polynomials (Extended DMD) (Williams *et al.* '15, 2000+ citations)

Most theoretical study of (E)DMD has been assuming $\mathcal{B} = L^2$, which gives just continuous spectrum for chaotic system.

Application 2: Koopman operators

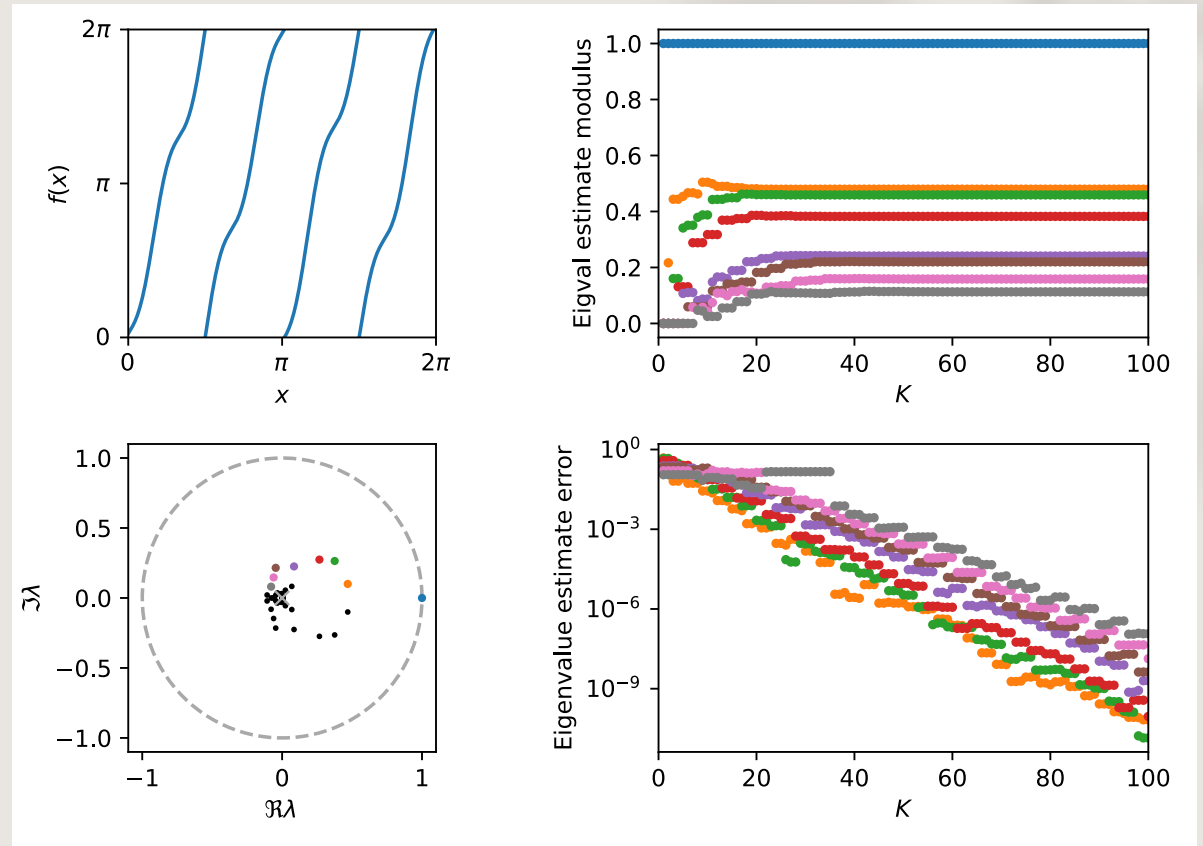
Theorem (W. '25): the operator error of trigonometric least squares approximation against a general sampling density ρ

$$\mathcal{P}\varphi = \operatorname{argmin}_{\substack{p \text{ deg. } \leq n \\ \text{trig. poly.}}} \int_0^{2\pi} |\varphi(x) - p(x)|^2 \rho(x) dx$$

is as small as that of Fourier projection (up to a constant).



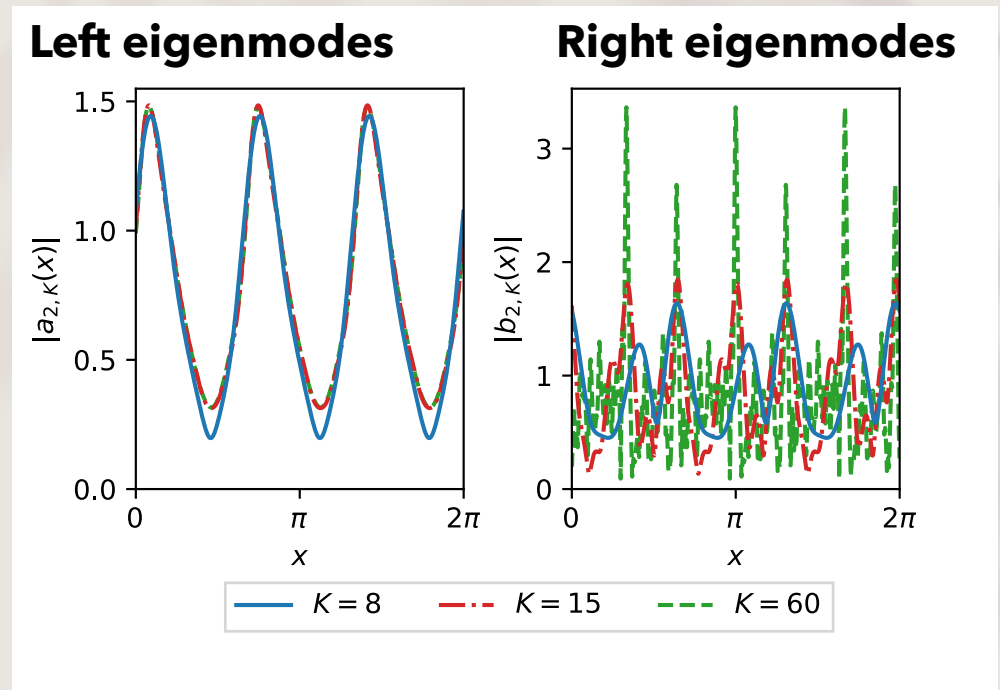
For *expanding* maps (i.e. the opposite of contractions), EDMD recovers **the discrete spectrum** of the Koopman operator...



Application 2: Koopman operators

...in a space of negative differentiability!

Q: how do data sampling errors behave in a space of negative differentiability?



...in fact, in the space $H(U)^*$

Outlook/bigger picture

- Transfer operators are an abstract, versatile way to study dynamics and their long-term behaviour
- We have methods to compute them rigorously and accurately (\Rightarrow reliably), in an increasingly large array of settings
- The beginning of some very long journeys!
 - Can we push on to study “difficult” cases like non-conformal fractals, non-uniformly hyperbolic dynamics?
 - Effectiveness of Koopman operator approximation from data? Can we quantify our uncertainty?

PhD scholarship on Koopman numerics from mid-2026!