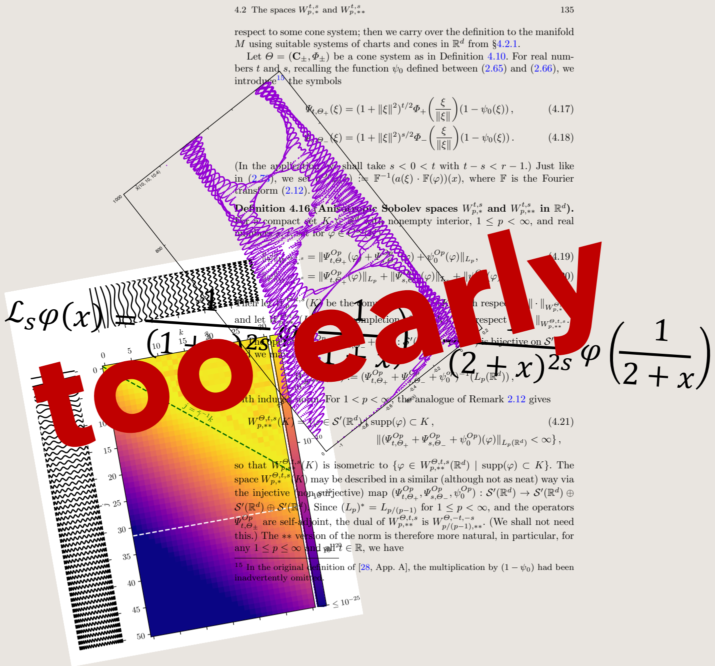
Computations with transfer operators

Caroline Wormell

The University of Sydney



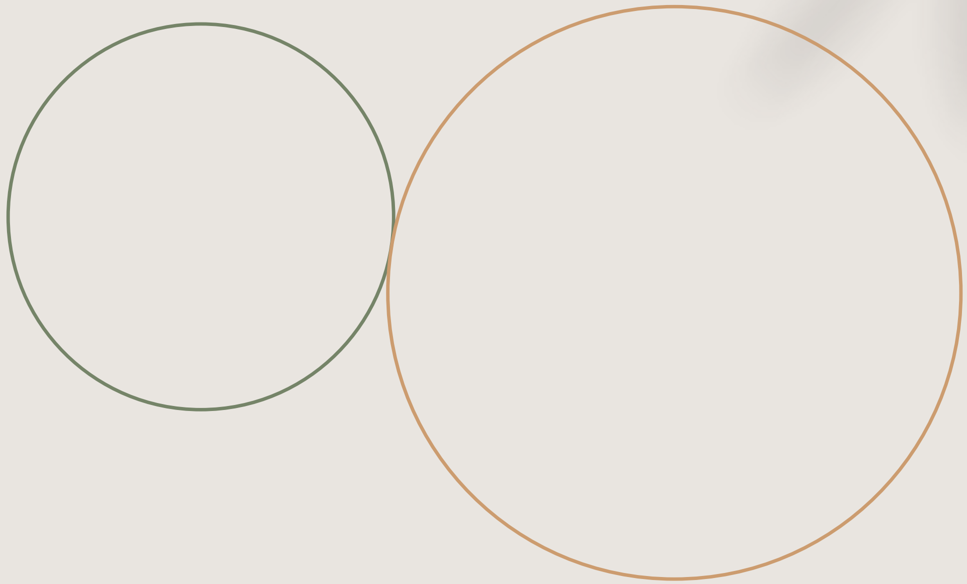
Circle geometry 😊

Circles

Theorem: it is possible to draw 1 circle

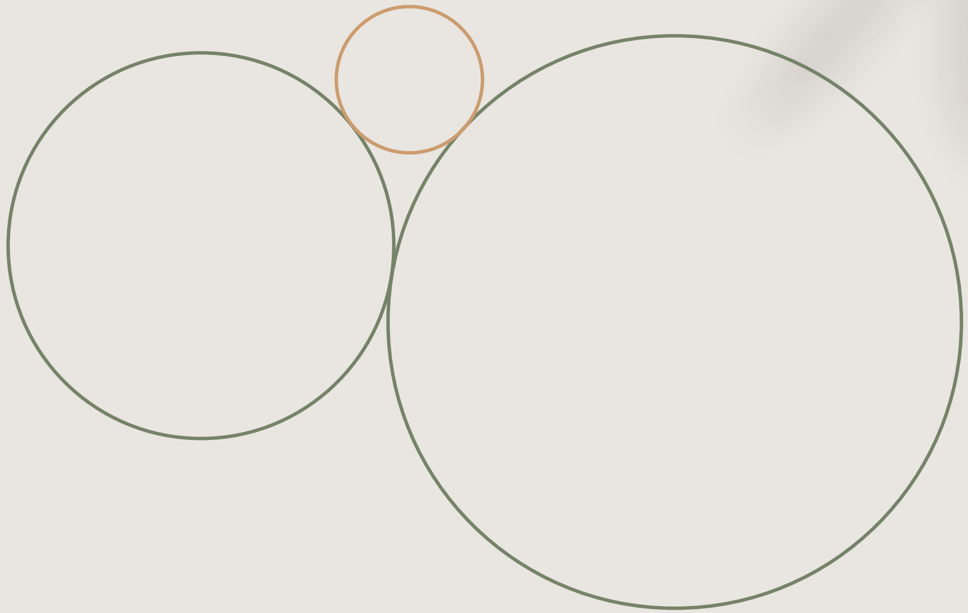
Circles

Theorem: Given any circle, it is possible to draw another one tangent to it



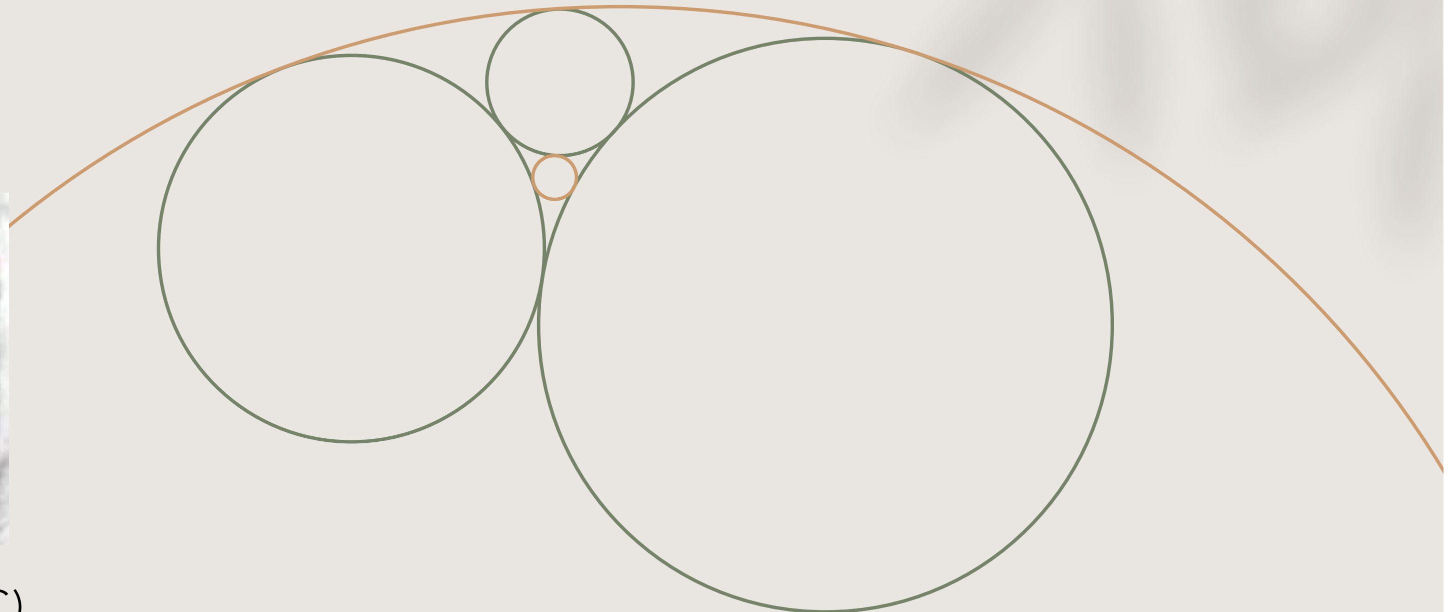
Circles

Theorem: Given any 2 tangent circles, it is possible to draw another one tangent to them both



Circles

Theorem (Apollonius): Given any 3 tangent circles, it is possible to draw another one tangent to them all.
There are two choices for adding the fourth one.



Apollonius of Perga (c. 240–c. 190 BC)
was the inventor of conics

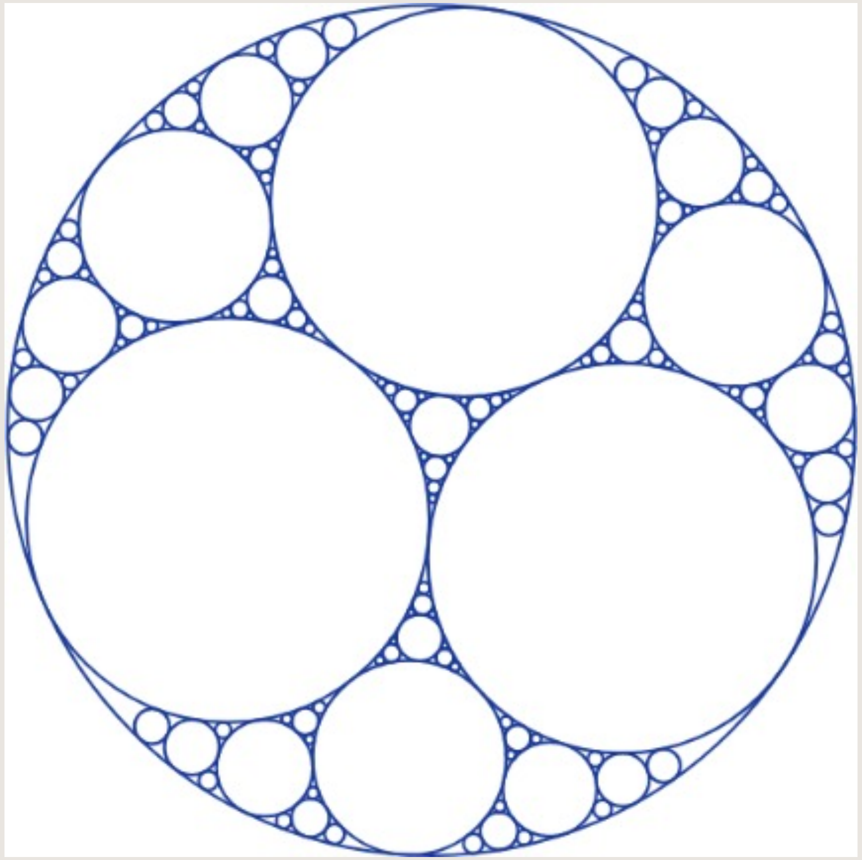
Circle packings

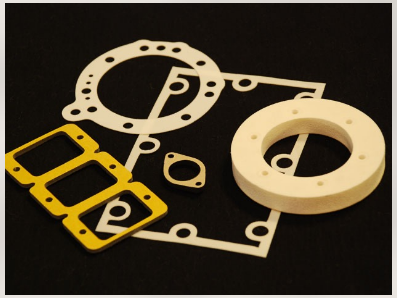
If you are bored, you can keep filling the gaps between the circles with other tangent circles:

René Descartes
(1596–1650)

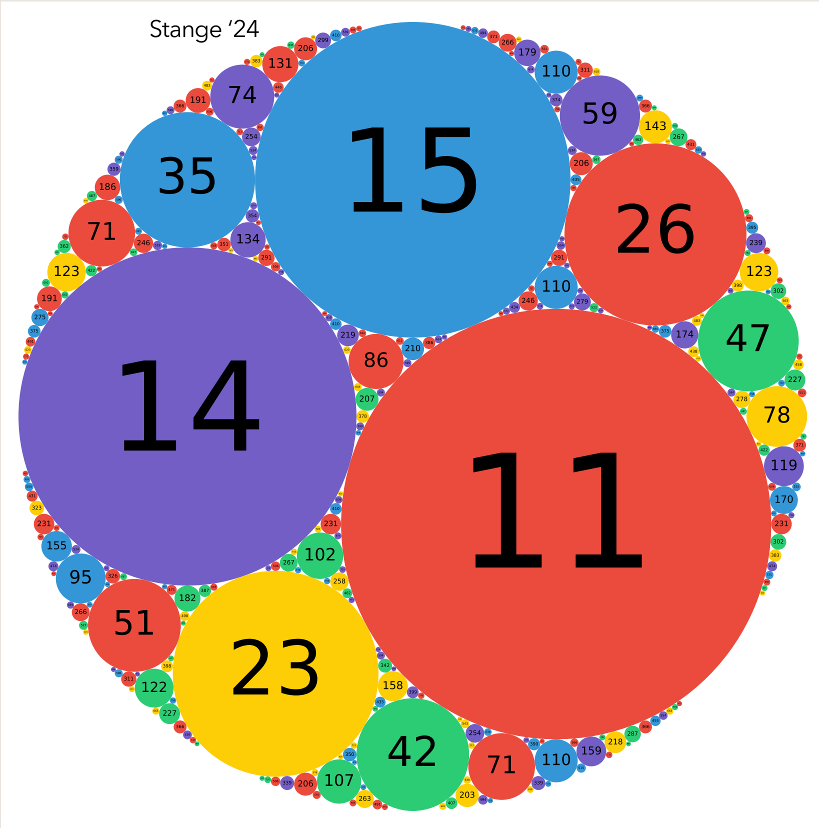
Seen in Smith's Alternative, Canberra

This gives you an Apollonian gasket





This gives you an Apollonian gasket



Outer circle = 6

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2)$$

Fact: if your first four circles have integer curvatures (=1/radius), so do the rest.

<u>*Counting circles*</u> leads to a lot of interesting number theory…

e.g. Curvatures must have certain sets of values mod 24 (Graham *et al.* '03)…

residues
0, 1, 4, 9, 12, 16
0, 5, 8, 12, 20, 21
0, 4, 12, 13, 16, 21
0, 8, 9, 12, 17, 20
3, 6, 7, 10, 15, 18, 19, 22
2, 3, 6, 11, 14, 15, 18, 23

with extra prohibitions on certain curvatures of the form αn^2, αn^4. (Haag *et al.* '24)

Counting circles

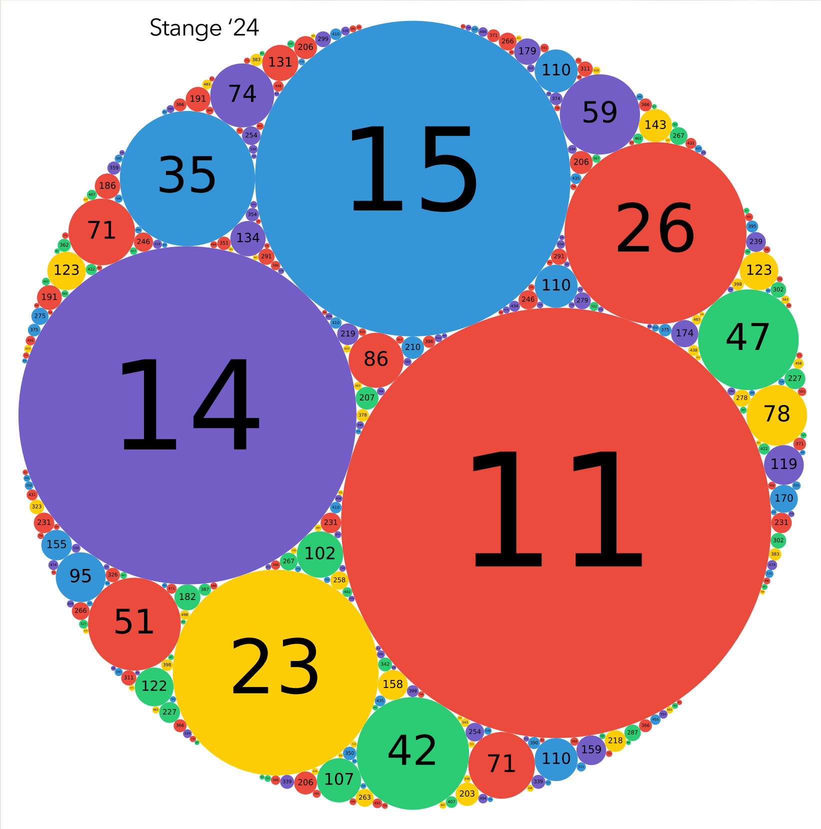
How many circles larger than a given size are there?

Theorem (Li–Oh, 2012)
For any bounded Apollonian packing,
$$(\#\text{circles of radius} > 1/T) \sim CT^{\alpha} + \mathcal{O}(T^{\beta}),$$
where α is the (universal) Hausdorff dimension of the gasket and $\beta < \alpha$.

No analytic form known for α…

…how can we find it?

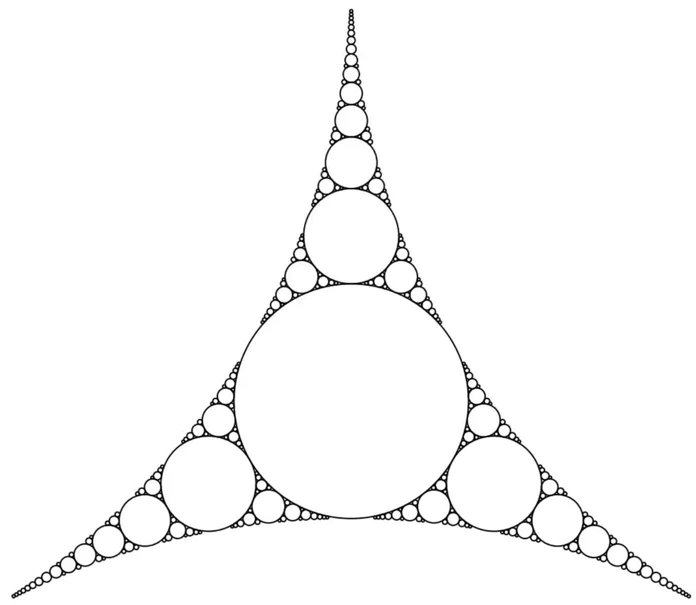
Computation



Key fact for computing:

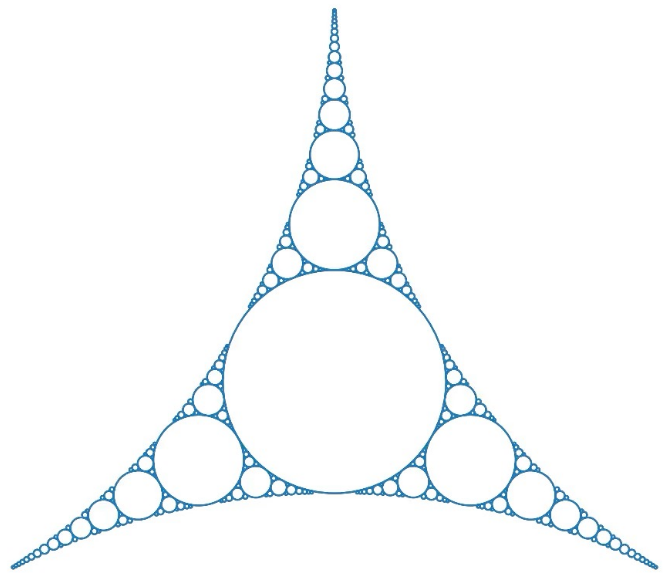
Apollonian gaskets considered as subsets of $\mathbb{C}$ are related by Möbius transformations

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C}$$



Key fact for computing

We can map the whole circular triangle
into sub-circular triangles.



So the Apollonian gasket is an *invariant set* under some *maps*.

Dynamics!!

Q: How do we pin down "average" limiting behaviour associated to some nonlinear transformations?

A: Using **transfer operators**

Topic of this talk

A *transfer operator* is a weighted dynamical composition operator:

$$(\mathcal{L}\varphi)(x) = \sum_{i \in I} w_i(x)\varphi(T_i(x))$$

function $\varphi: X \to \mathbb{C}$

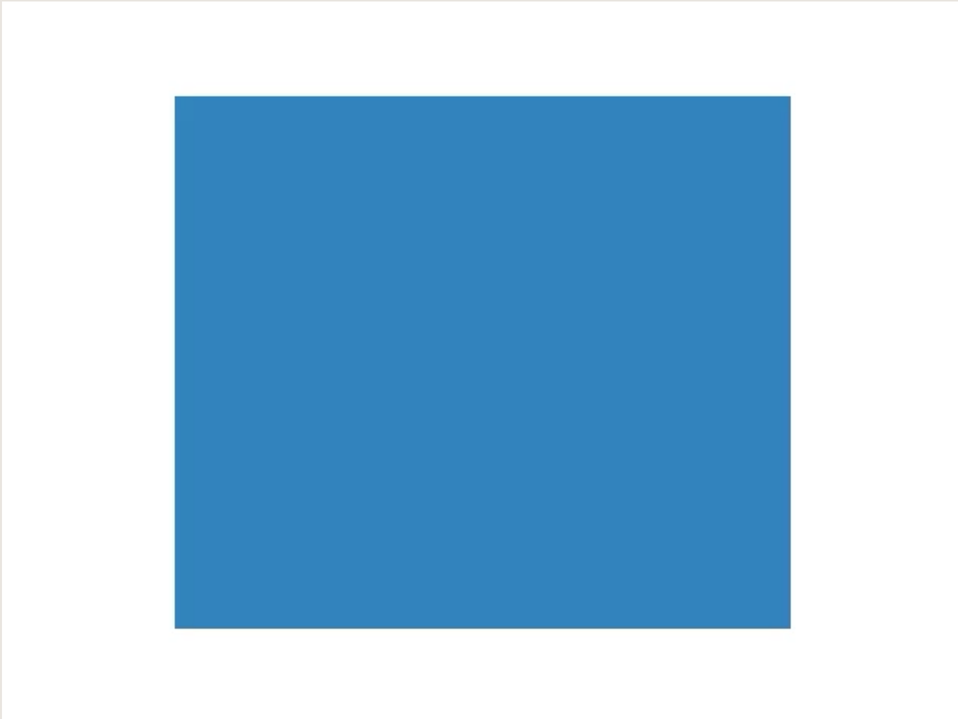
weights $w_i: X \to \mathbb{C}$
maps $T_i: X \to X$

This talk:

- Why transfer operators?
- How can you *compute* their properties with a high degree of certainty (rigour, accuracy)?
- What are some applications?

Why transfer operators?

The most obvious place to start in dynamics is in studying *invariant sets*.



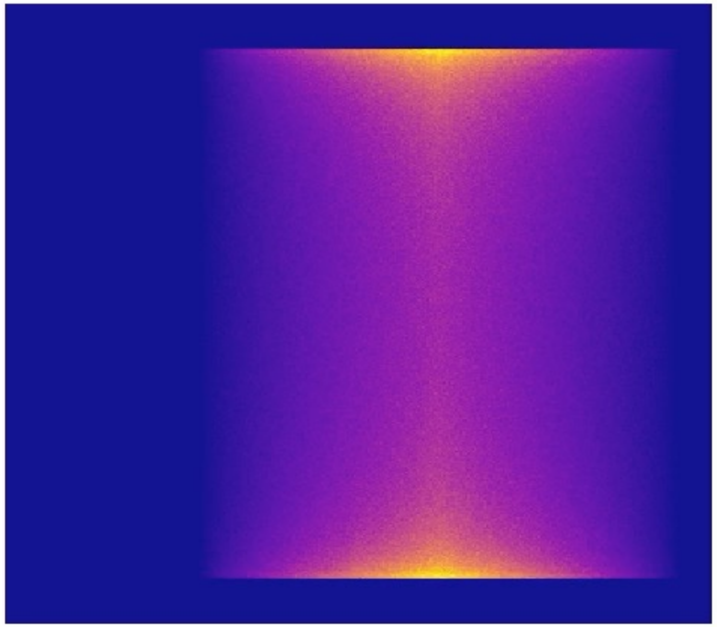
These aren’t linear objects, but signed measures on them are…

Transfer operator

Given transformations (say $T_i: X \to X$), we could define an operator that:

- divides up measures on X according to some functions $w_i > 0$ and
- pushes them forward by different transformations:

$$\mathcal{L}^* \mu \coloneqq \sum_{i \in I} (T_i)_* (w_i \mu)$$



Consider an eigenmeasure ν with support Λ:

$$\mathcal{L}^* \nu = \lambda \nu$$

Now

$$\Lambda = \operatorname{supp} \nu = \operatorname{supp} \mathcal{L}^* \nu = \cup_i T_i(\Lambda)$$

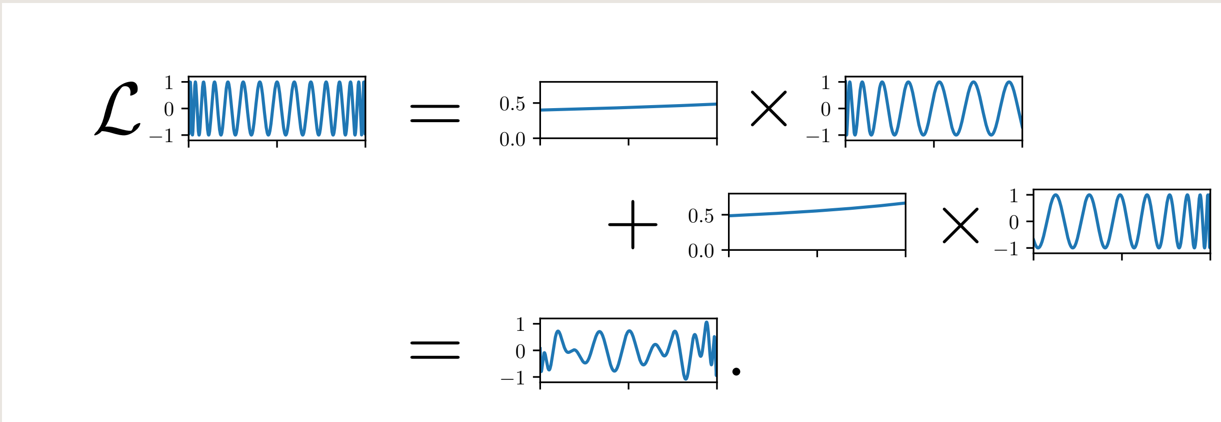
so Λ is an invariant set!

Duals

Studying the space of measures is a bit nasty, so we study the dual operator $\mathcal{L}: L^\infty(X) \to L^\infty(X)$

$$\mathcal{L}^*\mu \coloneqq \sum_i T_{i_*}(w_i \mu) \qquad \leftrightarrow \qquad \mathcal{L}\varphi = \sum_i w_i \varphi \circ T_i$$

This is much nicer as we are now dealing with functions!



Computation

- Most transfer operators don't produce analytic solutions (one exception: Blaschke products–see Cecilia's talk)
- To approximate them on a computer, we need:
  1. A Banach space where our transfer operator has some compactness properties ($\Longrightarrow$ stability under discretisation) **Difficult but a big industry**
  2. A sequence of finite-rank projections that converge quickly (i.e. a good discretisation) **In progress**
  3. A way to compute the action of the operator effectively. **Mostly(!) OK**
  4. Optionally, a way to *rigorously validate* your estimate. **Surprisingly good**

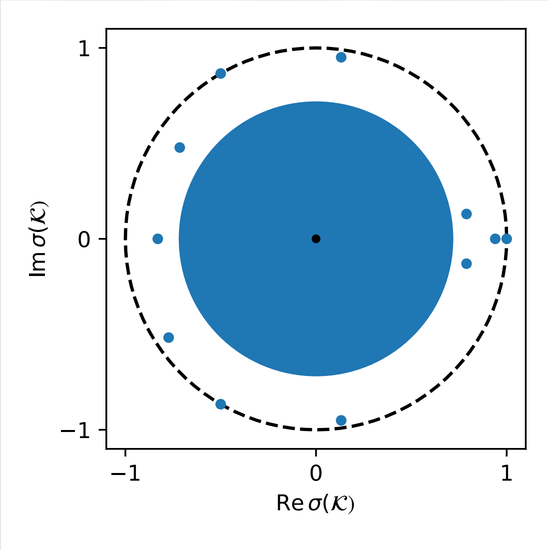
1. Stability to approximation

Many questions involving transfer operators (e.g. fractal dimension) concern an (often leading) eigenvalue.

Note:

- Point spectrum is much easier to approximate than continuous spectrum.
  Work on continuous spectrum: Colbrook *et al.* '23 and descendents…
- However, we know that the discrete spectrum is (mostly) consistent across Banach spaces (Baladi and Tsuji '16)

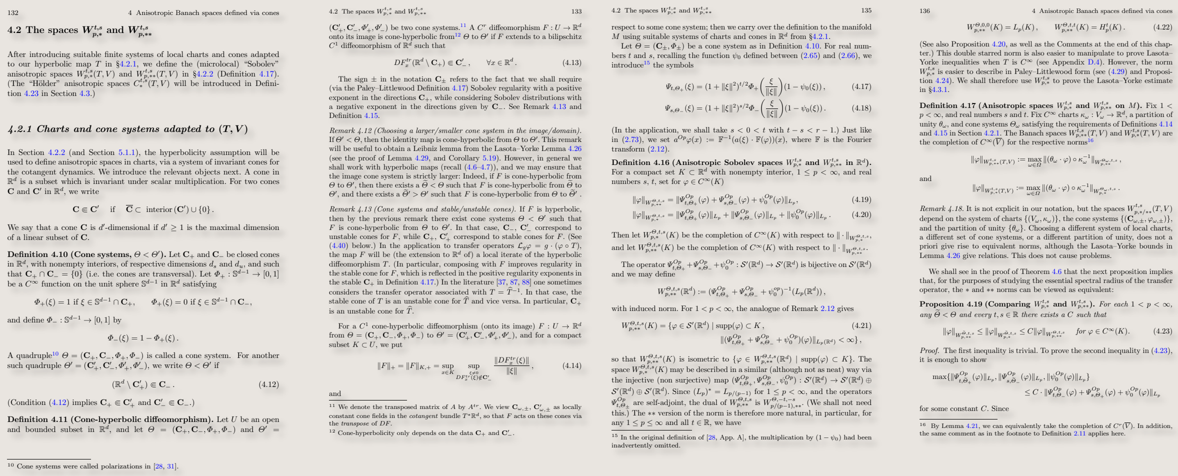
Goal: find Banach space $\mathcal{B}$ so that $\mathcal{L}: \mathcal{B} \to \mathcal{B}$ has discrete spectrum (at least somewhere).
"quasicompactness"



1. Stability to approximation

Goal: find a Banach space so that $\mathcal{L}: \mathcal{B} \to \mathcal{B}$ has discrete spectrum (at least where we are looking at it).

- For general maps T_i the Banach space $\mathcal{B}$ needs to be tailored to the dynamics (e.g. functions with regularity along unstable manifolds, etc.)



Some Banach spaces for uniformly hyperbolic dynamics (Baladi '18)

1. Stability to approximation

Note Apollonian circle packing is 2 real dimensions

Super ezy mode: uniform **1D** contractions:

$$T_i: [-1,1] \to [-1,1] \ s.t. \sup_{\substack{x \in [-1,1] \\ i \in I}} |T_i'(x)| \leq \gamma < 1.$$

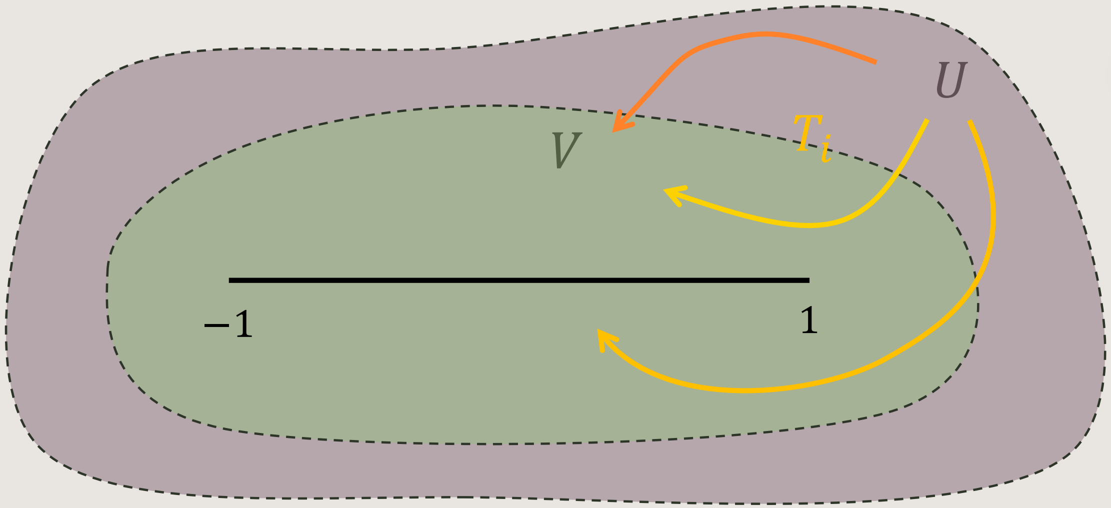
- Almost any space of at least Hölder functions will give us a quasi-compact operator.
- But we want the best for computation (="most compact")

1. Stability to approximation: Banach space

Let's imagine that there are some open sets U, V with

$$[-1,1] \subset U \subseteq \overline{U} \subset V \subseteq \mathbb{C}$$

so that the T_i, w_i extend analytically to U so each T_i maps U into V.



1. Stability to approximation: Banach space

Define the Hardy space:

$$H(U) = \{\varphi : U \to \mathbb{C} \text{ analytic and bounded}\}$$

with the sup-norm on U.

$$\varphi(z) = \sum_i w_i(z)\varphi\big(T_i(z)\big)$$

Then:

- $\mathcal{L}: H(V) \to H(U)$ is bounded
- $H(V)$ embeds compactly in $H(U)$.

So $\mathcal{L}: H(U) \to H(U)$ is compact and thus approximable in operator norm by finite rank operators

2. Discretisation: polynomials

Let's try and project our operator $\mathcal{L}$ onto a space of polynomials:

$$V = \{\text{polynomials of degree} \leq K\}$$

There are many ways to do this of varying quality.

Given K points $\{x_j\}$ we can choose as a basis of V the *Lagrange polynomials*, defined by:

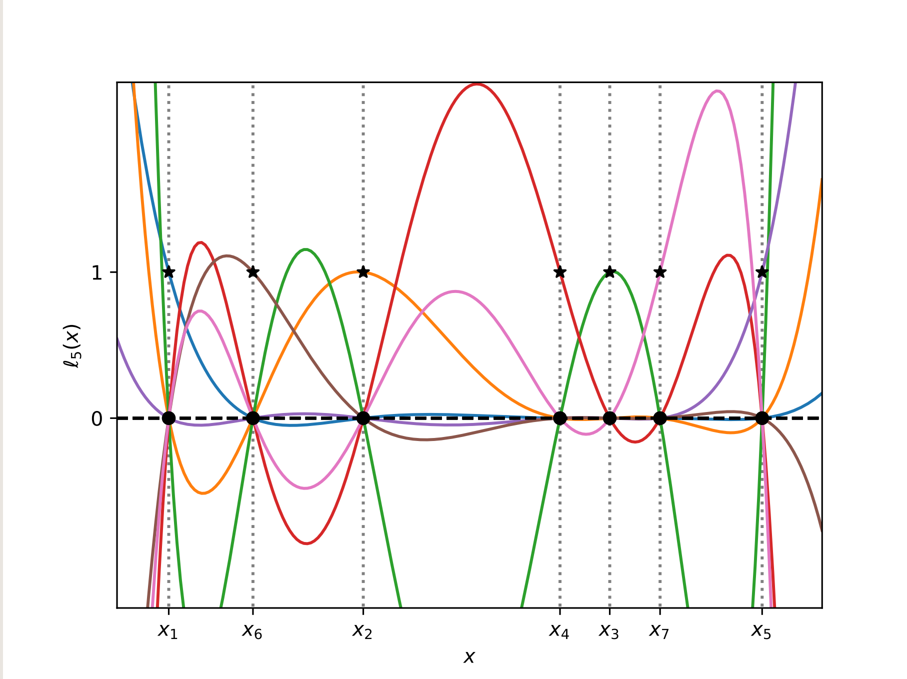
$$\ell_j(x_{j'}) = \begin{cases} 1, & j = j' \\ 0, & \text{else} \end{cases}$$

(so $\ell_j(x) = \prod_{j' \neq j} \frac{x - x_{j'}}{x_j - x_{j'}}$)

2. Discretisation: polynomials

We can use Lagrange polynomials to interpolate a function:

$$\varphi(x) \approx \sum_{j=1}^{K} \varphi(x_j)\ell_j(x)$$



For most choices of points $\{x_j\}$, this is a terrible idea.

2. Discretisation: polynomials

But for Chebyshev points, it works great:

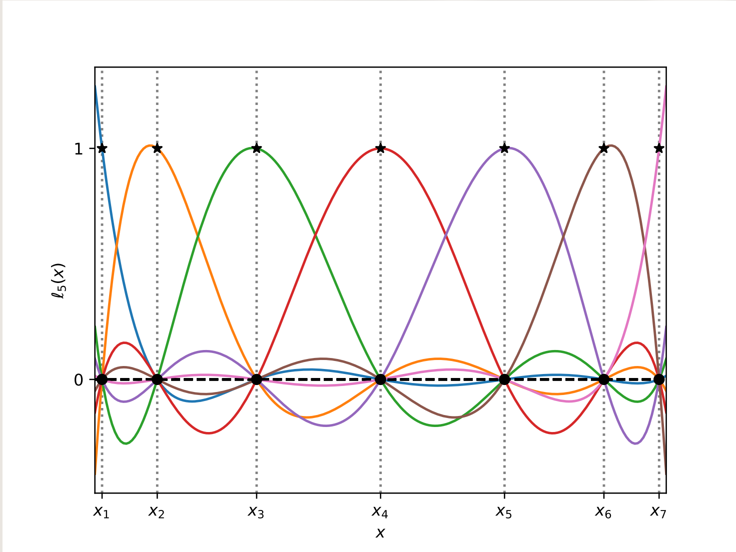
$$x_k = \cos\frac{\pi(2k-1)}{2k} \in [-1,1]$$

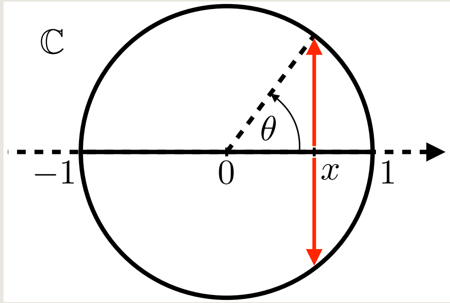
Why?

$$x \to \cos\theta$$

$$\text{Chebyshev points} \to \text{evenly spaced points}$$

$$\text{polynomials} \to \text{trig functions}$$



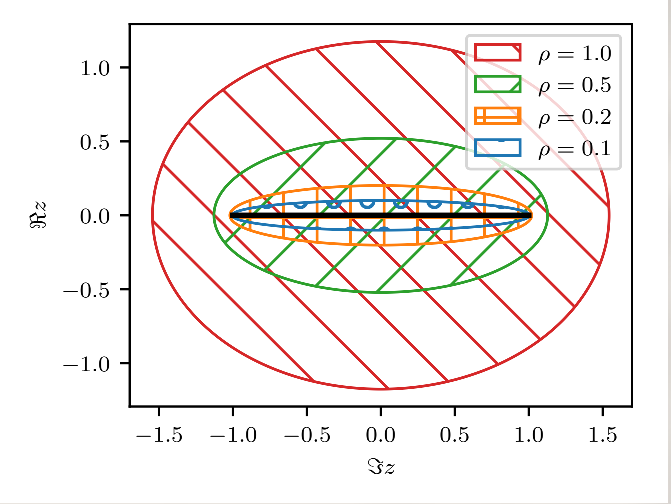


2. Discretisation: polynomials

We can interpolate $\mathcal{L}$ in the polynomial basis:

$$\mathcal{L}^{(K)}\ell_k = \sum_{j=1}^{K} \ell_j \, (\mathcal{L}\ell_k)(x_j)$$

Now, suppose our sets U, V are *Bernstein ellipses*:



$\cos(\mathbb{R} + i[-\rho, \rho\,])$

> **Theorem (Bandtlow & Slipantschuk '19, 1D; W. and Vytnova '25 nD)**
> Let $\mathcal{L}^{(K)}$ be the K-point interpolation of $\mathcal{L}$. Then there exist (constructible) constants C, c
> $$\left\|\mathcal{L}^{(K)} - \mathcal{L}\right\|_{H(U)} \leq C e^{-cK}$$

All our estimates are exponentially good!

Work using Chebyshev discretisations

- Lyapunov exponents (**W. ’19**, W. ’21, Pollicott–Vytnova ’23), statistical laws (W. ’19, Crimmins–Froyland ’19), metric entropy (Pollicott–Slipantschuk ’24)
- Eigenvalues, almost-invariant sets (**Bandtlow–Slipantschuk ’19**, Blumenthal *et al*. ’25)
- Hausdorff dimension (Pollicott–Vytnova ’22 , **Vytnova–W. ’25**)
- Selberg zeta functions (Bandtlow *et al*. ’21)
- Lagrange and Markov spectra (Pollicott–Vytnova ’22, Matheus–Moreira–Vytnova ’22)
- Fourier transform of fractal measures (W. ’23)
- Linear response problems (Nisoli–Taylor-Crush ’23, Froyland–Galatolo ’23)
- Extended Dynamical Mode Decomposition (Bandtlow–Just–Slipantschuk group ’23–, W. 25, Herwig *et al*. ’25, …)

Application 1: fractal Hausdorff dimension

If there exists a *Frostman measure* μ supported on a set Λ with

$$\mu\big(B(x,r)\big) \leq Cr^s \text{ for all } x \in \Lambda$$

then the Hausdorff dimension of Λ is greater than or equal to s.

Under some nice conditions (T_i conformal…) we can find such measures as eigenmeasures of a transfer operator ($\mathcal{L}_s^*\mu = \mu$)

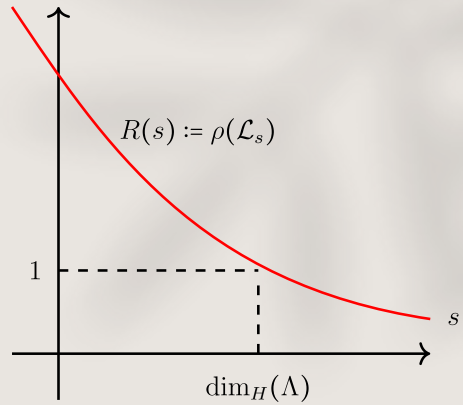
$$(\mathcal{L}_s\psi)(x) := \sum_{i\in I} |DT_i(x)|^s \psi(T_i(x))$$

Application 1: fractal Hausdorff dimension

Suppose the fractal Λ is generated by uniform conformal, non-overlapping contractions $\{T_i\}$. Define

$$(\mathcal{L}_s\psi)(x) := \sum_{i\in I} \underbrace{|DT_i(x)|^s}_{<1} \psi(T_i(x))$$

- $R(s) \coloneqq \operatorname{specrad} \mathcal{L}_s$ is strictly decreasing
- $R(\dim_H(\Lambda)) = 1$ (Ruelle–Bowen formula)



Application 1: fractal Hausdorff dimension

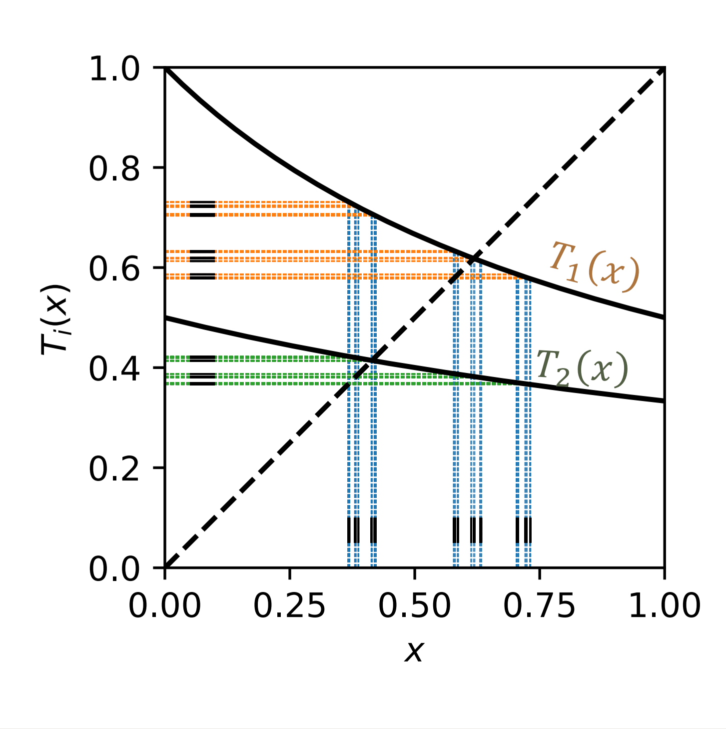
Numbers can have interesting *continued fractions.*

For example, they can just contain 1 and 2, e.g.

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cdots}}}}}$$

These numbers form a set $\mathrm{E}_2 \subset [0,1]$ invariant under the contractions*

$$T_1(x) = \frac{1}{1+x}, T_2(x) = \frac{1}{2+x}$$



Application 1: fractal Hausdorff dimension

The issue is basically to find $s \in [0,1]$ such that 1 is the leading eigenvalue of

$$\mathcal{L}_s\varphi(x) = \frac{1}{(1+x)^{2s}}\,\varphi\left(\frac{1}{1+x}\right) + \frac{1}{(2+x)^{2s}}\,\varphi\left(\frac{1}{2+x}\right)$$

Theorem 1.5. [3]

$$\begin{aligned}\dim_H(E_2) &= 0.5312805062\,7720514162\,4468647368\,4717854930\,5910901839\,8779888397\\ &\quad 8039275295\,3564383134\,5918109570\,1811852398\,8042805724\,3075187633\\ &\quad 4223893394\,8082230901\,7869596532\,8712235464\,2997948966\,3784033728\\ &\quad 7630454110\,1508045191\,3969768071\,3 \pm 10^{-201}.\end{aligned}$$

Details on the proof of this bound appear in §4.1.2. Whereas it may not be clear why a knowledge of $\dim_H(E_2)$ to 200 decimal places is beneficial, it at least serves to illustrate the effectiveness of the method we are using compared with earlier approaches.

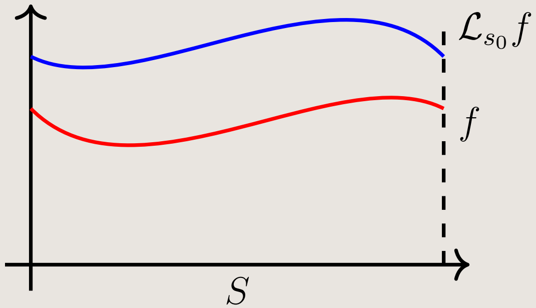
Pollicott and Vytnova, TAMS, 2022

More advanced applications to Markov and Lagrange spectra, Zaremba conjecture…

Rigorous validation

To turn your computer bound into a theorem, you need:

- Interval arithmetic to keep track of round-off errors and 1D approximations (e.g. truncating sums)
- A trick to go from finite dimensions to the full problem
  I used to loathe this part, but if you have the *right trick* it is not that bad
  The innovation of Pollicott–Vytnova '22 was to harness the fact that $\mathcal{L}_s$ are positive operators



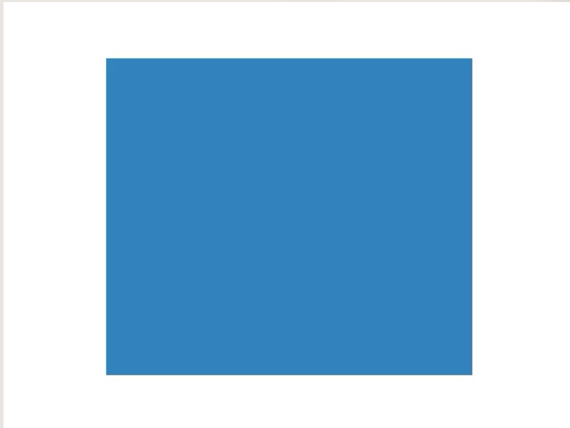
Application 1a: Apollonian circle packing

- For half a century the bound for the Apollonian gasket's dimension was:

> **Theorem** (Boyd, 1973):
> $$1.300197 < \alpha < 1.314534$$

- Various non-rigorous estimates using Ruelle–Bowen formula:
  Thomas and Dhar '94, Curtis McMullen '98, de Leo '14, Bai–Finch '18
  Issues:
  - The contractions are actually non-uniform
  - They didn't have much control over their discretisations

Joint work with Polina Vytnova (University of Surrey)



Application 1a: Apollonian circle packing

- For half a century the bound for the Apollonian gasket's dimension was:

Theorem (Boyd, 1973):

$$1.300197 < \alpha < 1.314534$$

- Various non-rigorous estimates using Ruelle–Bowen formula:
  Thomas and Dhar '94, Curtis McMullen '98, de Leo '14, Bai–Finch '18
  Issues:
  - The contractions are actually non-uniform
  - They didn't have much control over their discretisations

Joint work with Polina Vytnova (University of Surrey)

Non-uniform contractions

Our strategy* is to combine maps to get uniform contractions.

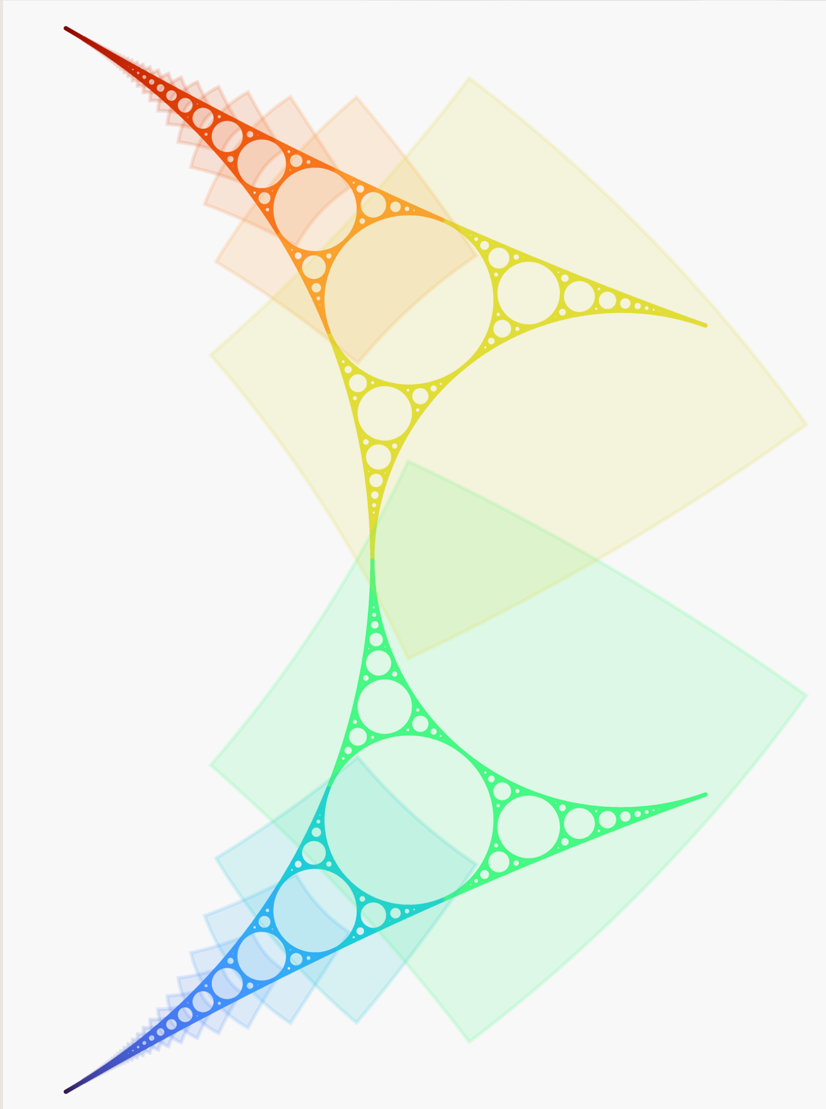
This led to an infinite number of branches:

$$(\mathcal{L}_s\varphi)(x) = \sum_{\substack{n=1\\ \pm\in\{+,-\}}}^{\infty} \underbrace{J_{n,\pm}(x)^s\varphi(T_{n,\pm}(x))}_{O(n^{-2s})}$$

But we can approximate the tail via Euler–Maclaurin formula…

$$\sum_{n=N}^{\infty} J_{n,\pm}(x)^s\varphi(T_{n,\pm}(x)) \sim \sum_{l=0}^{\infty} c(l,s)N^{1-2s-l}$$

* probably the most computationally effective approach for any low-dimensional dynamics ⁉️



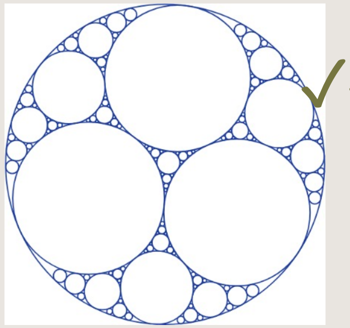
Application 1a: Apollonian circle packing

Problem of computation "solved":

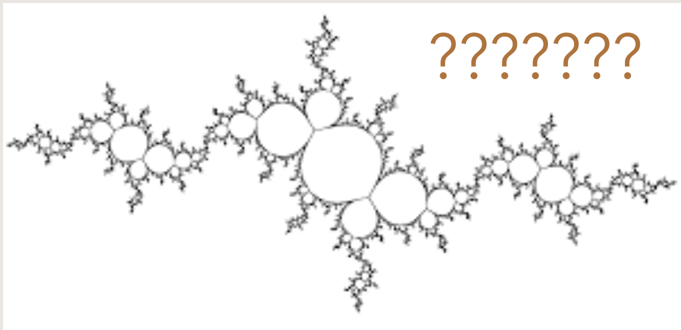
Theorem (Vytnova and W., *Invent. Math.* 2025)

$$\alpha = 1.3056867280\ 4987718464\ 5986206851\ 0408911060\ 2644149646\ 8296446188\ 3889969864\ 2050296986\ 4545216123\ 1505387132\ 8079246688\ 2421869101\ 967305643 \pm 10^{-129}$$

- In this algorithm, resources ~ (#certified digits)$^{5+\epsilon}$.
- The same ideas transfer to a lot of other fractals!

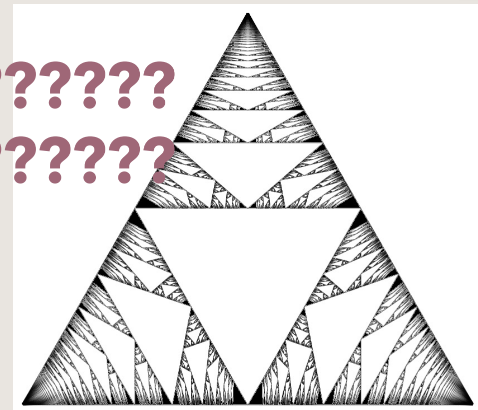


✓✓✓



Parabolic Julia sets

Rauzy gasket (from interval exchange transformations)



???????
???????

Application 2: Koopman operators

Suppose you have a dynamical system $T: M \to M$ and you are interested in the evolution of states (Fokker–Planck operator but deterministic).

That involves studying

$$\mathcal{K}^* \mu \coloneqq T_* \mu$$

or its adjoint, known as the Koopman operator:

$$\mathcal{K}\psi \coloneqq \psi \circ T$$

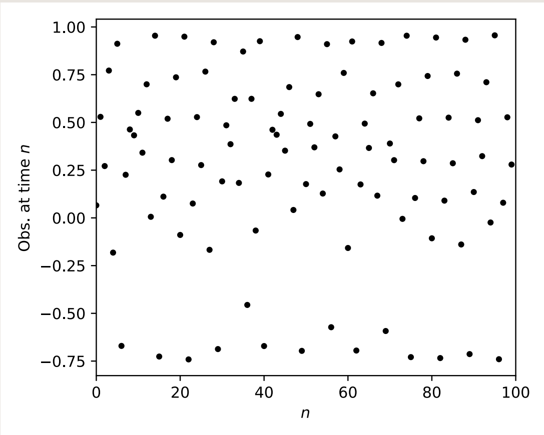
The eigenfunctions of this operator identify invariant sets, almost–invariant sets…

Application 2: Koopman operators

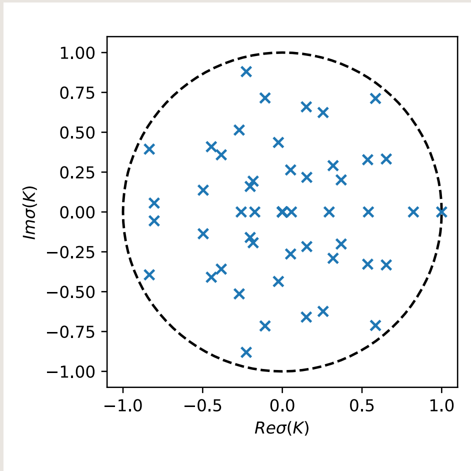
$$\mathcal{K}\psi \coloneqq \psi \circ T$$

Hope: study Koopman operators from observations:

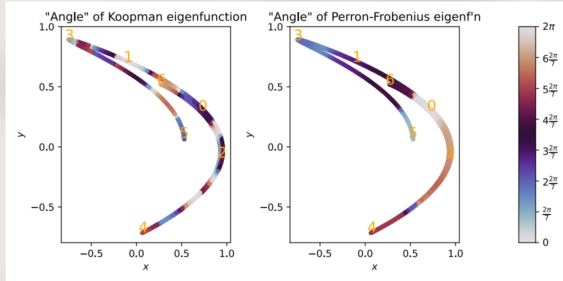
$$(\mathcal{K}\psi)(x_k) = T(x_k) = x_{k+1}$$



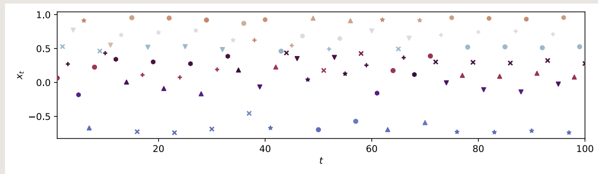
Time series (maybe high-dimensional/partially observed/…)

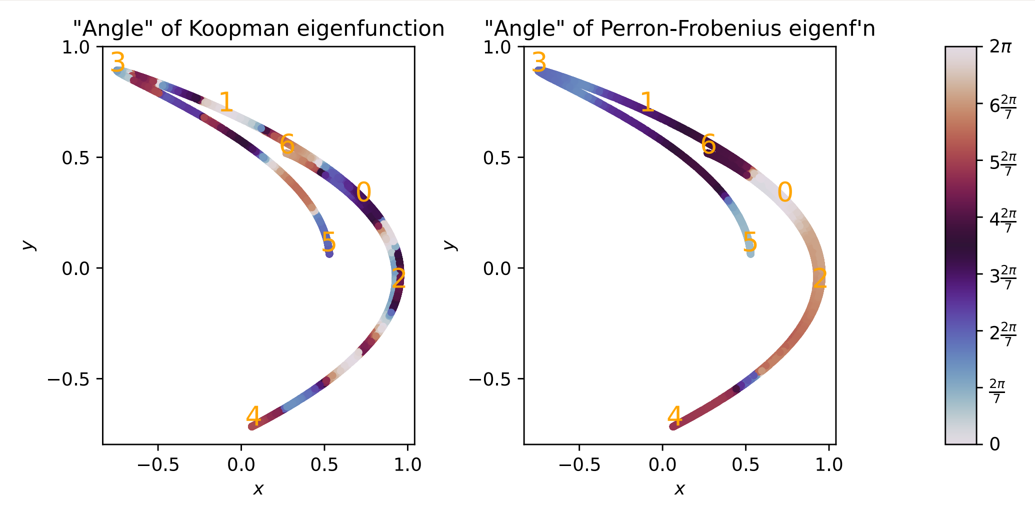


$\sigma(K)$

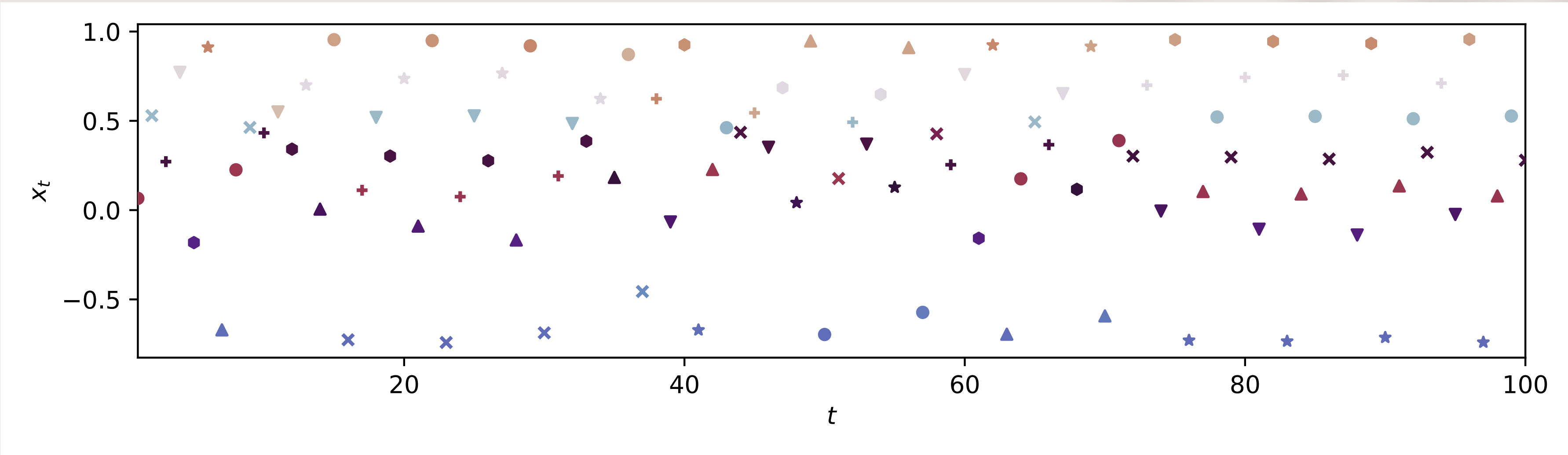


Almost invariant sets, metastable structures…





"Angle" of Koopman eigenfunction
"Angle" of Perron-Frobenius eigenf'n
x
y
2π
0



x_t
t

Application 2: Koopman operators

All our ideas are the same but with different basis functions.

Given function basis $\{\psi_0, \psi_1, \dots, \psi_B\}$ find matrix K that minimizes least squares error:

$$K\boldsymbol{v} = \underset{\substack{\text{p deg.} \leq n \\ \text{trig.poly.}}}{\operatorname{argmin}} \sum_{m=1}^{M} \left| \sum_b (K\boldsymbol{v})_b \psi_b(x_m) - \sum_b \boldsymbol{v}_b \underbrace{\psi_b(T(x_m))}_{\mathcal{K}\psi_b(x_m)} \right|^2$$

Application 2: Koopman operators

Huge industry, and it depends on your basis functions:

- piecewise constant functions (= Ulam's method) (Dellnitz *et al.* 2001 – 300+ citations)
- Via linear functions/delay variables (DMD) (Tu '13, 2600+ citations)
- Via low-order polynomials (Extended DMD) (Williams *et al.* '15, 2000+ citations)

Most theoretical study of (E)DMD has been assuming $\mathcal{B} = L^2$, which gives just continuous spectrum for chaotic system.

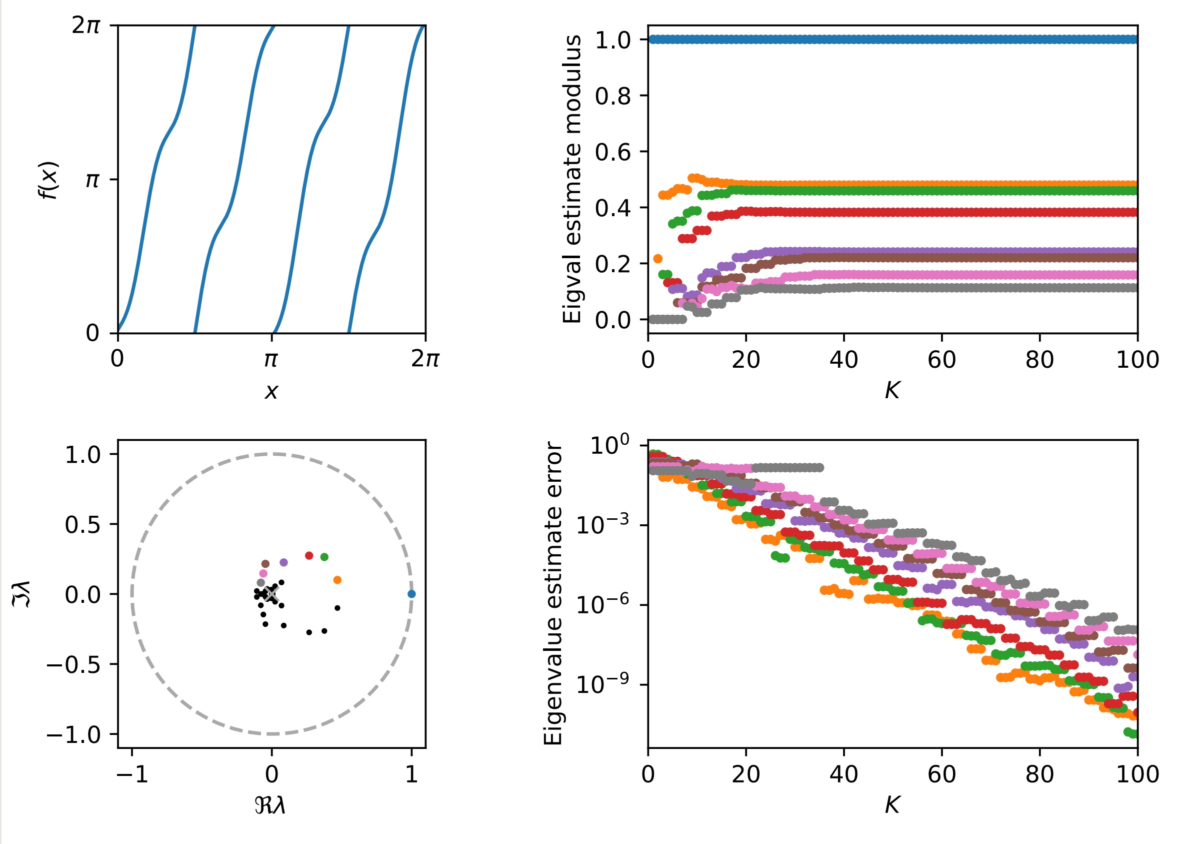
Application 2: Koopman operators

Theorem (W. '25): the operator error of trigonometric least squares approximation against a general sampling density ρ

$$\mathcal{P}\varphi = \underset{\substack{\text{p deg.} \leq n \\ \text{trig.poly.}}}{\operatorname{argmin}} \int_0^{2\pi} |\varphi(x) - p(x)|^2 \rho(x) \mathrm{d}x$$

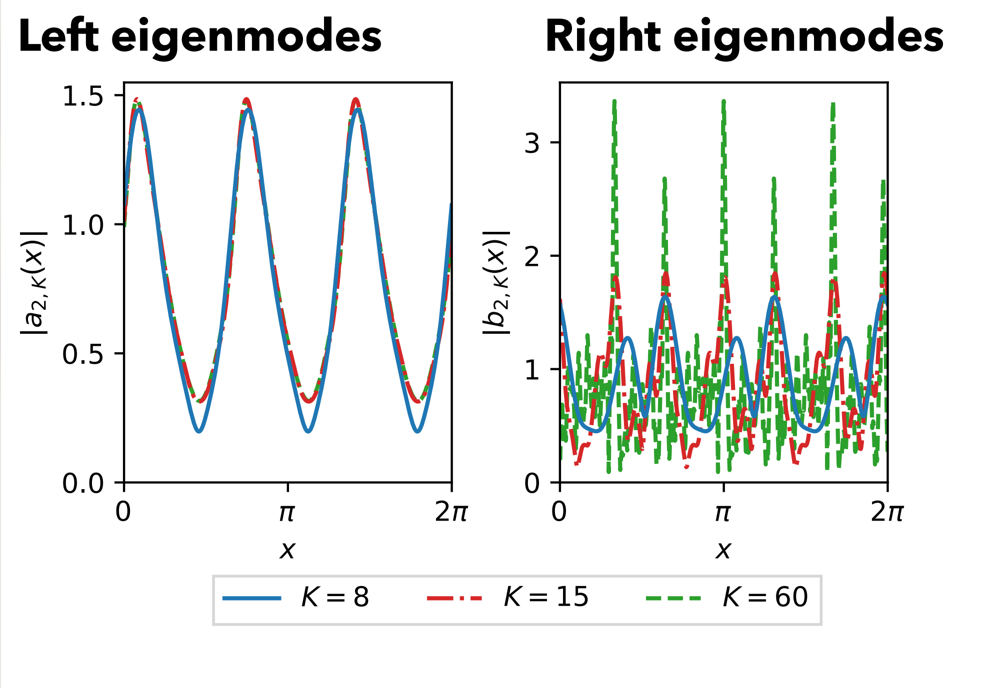
is as small as that of Fourier projection (up to a constant).

For *expanding* maps (i.e. the opposite of contractions), EDMD recovers **the discrete spectrum** of the Koopman operator…



Application 2: Koopman operators

…in a space of negative differentiability!



Q: how do data sampling errors behave in a space of negative differentiability?

…in fact, in the space $H(U)^*$

Outlook/bigger picture

- Transfer operators are an abstract, versatile way to study dynamics and their long-term behaviour
- We have methods to compute them rigorously and accurately ($\Longrightarrow$ reliably), in an increasingly large array of settings
- The beginning of some very long journeys!
  - Can we push on to study "difficult" cases like non-conformal fractals, non-uniformly hyperbolic dynamics?
  - Effectiveness of Koopman operator approximation from data? Can we quantify our uncertainty?

Advertisement x

PhD scholarship on Koopman numerics from mid-2026!